II. *De Seriebus infinitis Tractatus. Pars Prima.* *Auctore* Petro Remundo *de* Monmort. R. S. S.

Prop. 1. *Prob.*

INvenire ſummam terminorum quot libuerit Seriei hujus $a \times \overline{a+n} \times \overline{a+2n} \times \&c. \times \overline{a+p-1n}$ $+ \overline{a+n} \times \overline{a+2n} \times \overline{a+3n} \times \&c. \times \overline{a+pn}$ $+ \overline{a+2n} \times \overline{a+3n} \times \overline{a+4n} \times \&c. \times \overline{a+p+1n}$ $+ \overline{a+3n} \times \&c.$ Ubi eſt n differentia data, tam inter Factores continuos, a, $a+n$, $a+2n$, &c ejuſdem cujuſvis termini, quàm inter Factores homologos terminorum diverſorum in Serie continuatâ; atque deſignat p numerum factorum hujuſmodi in quovis termino.

Solutio Per x deſignetur primus Factorum in ultimo terminorum quorum ſumma requiritur, atque ſumma illa erit

$$\frac{x \times \overline{x+n} \times \&c. \times \overline{x+pn} - \overline{a-n} \times a \times \&c. \times \overline{a+p-1n}}{\overline{p+1n}}$$

Q. E. I.

Ex. 1. Proponatur Series numerorum naturalium $1+2+3+4+\&c.$ & invenienda ſit ſumma tot terminorum quot ſunt unitates in numero z, qui in hoc caſu eſt etiam ultimus terminorum quorum ſumma requiritur. In hoc itaque caſu ſunt $a=1$, $n=1$, $p=1$, & $x=z$. Unde fit $x \times \overline{x+n} \times \&c. \times \overline{x+pn} = z \times \overline{z+1}$, $\overline{a-n} \times a \times \&c. \times \overline{a+p-1n} = 0 \times 1$, atque $\overline{p+1n} = 2 \times 1$; adeoque ſumma quæſita eſt $\frac{z \times \overline{z+1}}{2}$.

Ex. 2. Invenienda ſit ſumma tot terminorum, quot ſunt unitates in numero z, Seriei $1+3+6+10+\&c.$ Numerorum Triangularium. Numeri 1, 3, 6, 10, &c. in hac

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Serie sic scribi possunt $\frac{1\times2}{2}$, $\frac{2\times3}{2}$, $\frac{3\times4}{2}$, $\frac{4\times5}{2}$, *&c.* Hoc pacto, seposito divisore dato 2, Series revocatur ad formam Propositionis, existentibus $a=1$, $n=1$, & $p=2$. $x=z$. Unde summa Seriei duplicata est $\frac{x\times\overline{x+1}\times\overline{x+2}-0\times1\times2}{3} = \frac{x\times\overline{x+1}\times\overline{x+2}}{3}$; adeoque habitâ ratione divisoris 2, Summa Seriei ipsius est $\frac{x\times\overline{x+1}\times\overline{x+2}}{2\times3}$, vel $\frac{z\times\overline{z+1}\times\overline{z+2}}{2\times3}$, in hoc casu existente x eodem ac z. Ad eundem modum inveniuntur summæ cæterorum numerorum figuratorum, quorum ormulæ jam vulgò innotescunt.

Ex. 3. Sint $a=1$, $n=2$, $p=3$, ut sit Series proposita $1\times3\times5+3\times5\times7+5\times7\times9+$ *&c.* In hoc itaque casu formula summæ fit $\frac{x\times\overline{x+2}\times\overline{x+4}\times\overline{x+6}-\overline{1-2}\times1\times3\times5}{4\times2} = \frac{x\times\overline{x+2}\times\overline{x+4}\times\overline{x+6}+15}{8}$. Verbi gratiâ, si quæratur summa decem terminorum, fit $x=19$ (nempe terminus decimus in Serie Arithmeticè proportionalium, 1, 3, 5, 7, *&c.*) adeoque summa est $\frac{19\times21\times23\times25+15}{8}$ $=28680$. Propositio vero sic demonstratur.

Demonstratio. Sit Series quantitatum A, B, C, D, E, *&c.* quarum differentiæ constituant Seriem a, b, c, d, *&c.* (nempe ut sint $a=B-A$, $b=C-B$, $c=D-C$, *&c.*) Hinc statim colligitur esse $a+b=C-A$, $a+b+c=D-A$, $a+b+c+d=E-A$: & in genere aggregatum quotlibet terminorum Seriei a, b, c, d, *&c.* æquale est termino proximè insequenti Seriei A, B, C, D, E, *&c.* mulctato termino primo A. Pro A, B, C, *&c.* sume terminos

$\frac{\overline{a-n}\times a\times \&c.\times \overline{a+\overline{p-1}\,n}}{\overline{p+1}\,n}$, $\frac{a\times \overline{a+n}\times \&c.\times \overline{a+pn}}{\overline{p+1}\,n}$, $\frac{\overline{a+n}\times \overline{a+2n}\times \&c.\times \overline{a+\overline{p+1}\,n}}{\overline{p+1}\,n}$, &c. hoc est, valores successivos ipsius $\frac{x\times \overline{x+n}\times \&c.\times \overline{x+pn}}{\overline{p+1}\,n}$; & eorum differentiæ, pro a, b, c, d, &c sumendæ, erunt $a\times\overline{a+n}\times\&c.\times\overline{a+\overline{p-1}\,n}$, $\overline{a+n}\times\overline{a+2n}\times\&c.\times\overline{a+pn}$, &c. qui sunt ipsissimi termini Seriei propositæ. Sed comparando has Series, si terminus aliquis Seriei posterioris sit $x\times\overline{x+n}\times\&c.\times\overline{x+\overline{p-1}\,n}$, constat terminum uno ulteriorem in Serie priori fore $\frac{x\times\overline{x+n}\times\&c.\times\overline{x+pn}}{\overline{p+1}\,n}$. Summa itaque Seriei posterioris usque terminum $x\times\overline{x+n}\times\&c.\times\overline{x+\overline{p-1}\,n}$ inclusivè est $\frac{x\times\overline{x+n}\times\&c.\times\overline{x+pn}-\overline{a-n}\times a\times\&c.\times\overline{a+\overline{p-1}\,n}}{\overline{p+1}\,n}$ *Q. E. D.*

Scholium 1. In hâc propositione continetur particula quædam Methodi incrementorum, de quâ ante biennium librum edidit D. *Brook Taylor Soc. Reg. Lond. Secr.* mihi amicitiâ conjunctissimus. Librum ipsum adeat qui de eâ methodo plura scire velit: ad institutum nostrum sufficit observare quanta intersit affinitas inter Methodum hanc & Methodum Fluxionum seu differentialem Nam ut in Methodo differentiali, ad inveniendum differentiale ipsius x dignitatis x^m, unum latus x convertendum est in differentiam dx; & ortum ducendum est in dignitatis Indicem m, ut sit $m\,dx\,x^{m-1}$ differentiale quæsitum; sic in Methodo Incrementorum *Ad inveniendum Incrementum facti hujusmodi* $x\times\overline{x+n}\times\overline{x+2n}$, (*ubi factores* x, $x+n$,

$x + 2n$, *sunt in progressione Arithmetica, cujus differentia communis est ipsius x Incrementum datum n,) Factorum minimus x convertendus est in Incrementum, & ortum ducendum est in numerum Factorum, ut sit* $3n \times \overline{x+n} \times \overline{x+2n}$ *Incrementum quæsitum, numero Factorum in casu exposito existente 3.* Sic etiam ipsius $x \times \overline{x+n}$ Incrementum fit $2n \times \overline{x+n}$.

2. Incrementa etiam Reciprocorum hujusmodi Factorum inveniuntur per eandem regulam; hoc nempe observato, quòd cum sit Divisio contrarium Multiplicationis, vice ablationis minimi Factorum, sit jam addendus alius factor adhuc uno Incremento major; item quòd Factorum numerus sit scribendus cum signo negativo. Hoc pacto ipsius $\frac{1}{x}$ Incrementum fit $\frac{-1 \times n}{x \times \overline{x+n}}$; ipsius $\frac{1}{x \times \overline{x+n}}$ Incrementum fit $\frac{-2 \times n}{x \times \overline{x+n} \times \overline{x+2n}}$; & sic de aliis hujusmodi. Hoc facile probatur sumendo differentias inter Integralium valores duos continuos.

3. Insistendo vestigiis Methodi directæ, hinc colliguntur præcepta Methodi inversæ, quibus inveniuntur Integralia Incrementorum oblatorum. *Applicetur enim Incrementum oblatum ad lateris Incrementum datum; addatur Factor adhuc uno Incremento minor, & applicetur ortum ad numerum Factorum sic auctorum.* Sic *e. g.* oblato Incremento $n \times x \times \overline{x+n} \times \overline{x+2n}$, fit primò $x \times \overline{x+n} \times \overline{x+2n}$; deinde $\overline{x-n} \times x \times \overline{x+n} \times \overline{x+2n}$, addito Factore $\overline{x-n}$; denique $\frac{\overline{x-n} \times x \times \overline{x+n} \times \overline{x+2n}}{4}$, quod est Integrale quæsitum. Hoc quidem ubi Factores sunt Multiplicantes; Ubi vero Factores occupant locum divisoris, mutatis mutandis, regula hæc est, *Applicetur Incrementum oblatum ad lateris incrementum datum; rejiciatur*

Factorum

Factorum maximus, & applicetur ortum ad numerum Factorum relictorum cum signo negativo. Exempli gratiâ oblato Incremento $\frac{n}{x \times \overline{x+n} \times \overline{x+2n}}$, fit primò $\frac{1}{x \times \overline{x+n} \times \overline{x+2n}}$, deinde $\frac{1}{x \times \overline{x+n}}$, denique $\frac{1}{-2 \times x \times \overline{x+n}}$, seu $\frac{-1}{2x \times \overline{x+n}}$, quod est Integrale quæsitum.

4. In casu hoc novissimo Integrale inventum, cum signo contrario, æquale est summæ omnium Incrementorum in Serie in infinitum continuatâ; *v. g.* est $\frac{1}{2x \times \overline{x+n}}$ $= \frac{n}{x \times \overline{x+n} \times \overline{x+2n}} + \frac{n}{\overline{x+n} \times \overline{x+2n} \times \overline{x+3n}}$ $+ \frac{n}{\overline{x+2n} \times \overline{x+3n} \times \overline{x+4n}} +$ *&c.* Nam in hoc casu, facto x tandem infinito, evanescit $\frac{1}{2x \times \overline{x+n}}$, hoc est, ultimus terminorum A, B, C; *&c.* fit nihil; & ob contrarietatem signorum Integralis & Incrementi, vice $-A$ exprimitur aggregatum per $+A$.

Lemma 1.

Per X designetur terminus quilibet in Serie quâvis numerorum M, N, O, P, *&c*; per x designetur locus termini istius X in Serie illâ (*v. g.* ut sit $x = 1$, quando designat X terminum primum M, sit $x = 2$, quando designat X terminum secundum N, & sic de cæteris) & sint terminorum M, N, O, P prima differentiarum primarum b, c prima differentiarum secundarum, d prima tertiarum, e prima quartarum, & sic porrò. Tum erit

$$X = M + b \times \frac{x-1}{1} + c \times \frac{x-1}{1} \times \frac{x-2}{2} + d \times \frac{x-1}{1} \times \frac{x-2}{2} \times \frac{x-3}{3} + e \times \frac{x-1}{1} \times \frac{x-2}{2} \times \frac{x-3}{3} \times \frac{x-4}{4} + \&c.$$

Sequitur hoc ex tabulâ æquationum *pag.* 66. tractatûs nostri *Essay d'Analyse, &c.*

Lemma 2.

Iisdem positis, per z designetur terminus quilibet in Serie Arithmeticè proportionalium a, $a+n$, $a+2n$, *&c.* & sit jam $X = A + Bz + Cz \times \overline{z+n} + Dz \times \overline{z+n} \times \overline{z+2n} + Ez \times \overline{z+n} \times \overline{z+2n} \times \overline{z+3n} + \&c.$ Tum ipsorum A, B, C, D, E, *&c.* valores erunt.

$$A = M + b \times \frac{-a}{n} + c \times \frac{-a}{n} \times \frac{-a-n}{2n} +$$

$$+ d \times \frac{-a}{n} \times \frac{-a-n}{2n} \times \frac{-a-2n}{3n} +$$

$$+ e \times \frac{-a}{n} \times \frac{-a-n}{2n} \times \frac{-a-2n}{3n} \times \frac{-a-3n}{4n} + \&c.$$

$$B = \frac{1}{n} \times b + c \times \frac{-a-n}{n} + d \times \frac{-a-n}{n} \times \frac{-a-2n}{2n}$$

$$+ e \times \frac{-a-n}{n} \times \frac{-a-2n}{2n} \times \frac{-a-3n}{3n} \ \&c.$$

$$C = \frac{1}{n} \times \frac{1}{2n} \times c + d \times \frac{-a-2n}{n} + e \times \frac{-a-2n}{n} \times \frac{-a-3n}{2n} + \&c.$$

$$D = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} d + e \times \frac{-a-3n}{n} + \&c.$$

$$E = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} \times \frac{1}{4n} e + \&c.$$

Ordo

Ordo formandi coefficientes ipsorum $b, c, d, e,$ &c. in his valoribus, per se est satis manifestus.

Demonstratio. Quoniam per x & z designantur termini correspondentes progressionum Arithmeticarum 1, 2, 3, 4, &c & $a, a+n, a+2n, a+3n,$ &c. indicabit $x-1$ numerum differentiarum n qui in z continetur, ut sit $z = a + \overline{x-1}\,n$. Hinc fit $x-1 = \frac{z-a}{n}$, $x-2 = \frac{z-n-a}{n}$, $x-3 = \frac{z-2n-a}{n}$, &c. Substituendo itaque hos valores $x-1$, $x-2$, $x-3$, &c. in Serie Lemmatis præcedentis, & terminis in ordinem redactis, prodeunt ipsorum $A, B, C,$ &c. valores exhibiti.

Cor. Ubi $a = n$, prodeunt $A, B, C, D,$ &c. per formulas simpliciores, nempe

$$A = M - b + c - d + e \text{ \&c.}$$

$$B = \frac{1}{n} \times \overline{b - 2c + 3d - 4e \text{ \&c.}}$$

$$C = \frac{1}{n} \times \frac{1}{2n} \times \overline{c - 3d + 6e \text{ \&c.}}$$

$$D = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} \times \overline{d + 4e \text{ \&c.}}$$

Lemma 3.

Symbolis X & x eodem modo interpretatis ac in Lemmate primo, sint $q, r, s, t, u,$ &c. generatores Trianguli Arithmetici cujus lineam transversam, occupat Series $M, N, O, P, Q,$ &c in ordine nempe inverso, ut sit $q\ (= M)$ generator ultimus, r penultimus, s antepenultimus, & sic porrò. Tum erit

$$X = q + r \times \frac{x-1}{1} + s \times \frac{x-1}{1} \times \frac{x}{2} + t \times \frac{x-1}{1} \times \frac{x}{2} \times \frac{x+1}{3}$$
$$+ \text{ \&c.}$$

Constat

Constat ex contemplatione ipsius Trianguli Arithmetici, quam exhibuimus *pag.* 63 tractatûs *Essay d'Analyse, &c.* ubi idem fusius explicatur.

Lemma 4.

Iisdem positis, & Symbolo z eodem modo interpretato ac in *Lem.* 2. si sit $X = A + Bz + Cz \times \overline{z+n} + \&c.$ ut in *Lem.* 2. erunt coefficientium $A, B, C, D, \&c.$ valores.

$$A = q + r \times \frac{-a}{n} + s \times \frac{-a}{n} \times \frac{-a+n}{2n}$$

$$+ t \times \frac{-a}{n} \times \frac{-a+n}{2n} \times \frac{-a+2n}{3n} + \&c.$$

$$B = \frac{1}{n} \times r + s \times \frac{-a}{n} + t \times \frac{-a}{a} \times \frac{-a+n}{2n} + \&c.$$

$$C = \frac{1}{n} \times \frac{1}{2n} \times s + t \times \frac{-a}{n} + \&c.$$

$$D = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} \times t + \&c.$$

Ordo coefficientium in his valoribus est manifestus, & demonstratur Lemma ad modum *Lemmatis* 2.

Cor. 1. Ubi $a = n$, coefficientes, $A, B, C, D, \&c.$ prodeunt per formulas simpliciores, nempe

$$A = q - r, \qquad C = \frac{1}{n} \times \frac{1}{2n} \times \overline{s - t}$$

$$B = \frac{1}{n} \times \overline{r - s}, \qquad D = \frac{1}{n} \times \frac{1}{2n} \times \frac{1}{3n} \overline{t - u} \qquad \&c.$$

Cor. 2. Unde si generatorum $q, r, s, t, u, \&c.$ aliquot sint inter se æquales, exhibebitur X per formulam simpliciorem, evanescentibus aliquot coefficientium $A, B, C, D, \&c.$

Sic

Sic exempli gratiâ, propositâ Serie numerorum 4, 69, 530, 2676, 10350, &c. qui constituunt lineam decimam transversam in Triangulo Arithmetico cujus generatores tres priores sunt 54, — 18, 5, & septem posteriores sunt æquales 4; existente $a = 1 = n$, Terminus X exhibetur per formulam quatuor tantùm terminorum.

$$-\frac{z}{1}.\frac{z+1}{2}.\frac{z+2}{3}\ \&c. \times\frac{z+6}{7}+23\,\frac{z}{1}.\frac{z+1}{2}.\&c.\times\frac{z+6}{7}-72\,\frac{z}{1}.\frac{z+1}{2}.\&c.\times\frac{z+7}{8}+54\,\frac{z}{1}.\frac{z+1}{2}.\&c.\times\frac{z+8}{9}.$$

evanescentibus coefficientibus sex primis A, B, C, D, E, F.

Prop. II. Prob.

Invenire summam quotlibet terminorum Seriei

$$\frac{M}{a\times\overline{a+n}\times\&c.\times\overline{a+p-1n}}+\frac{N}{\overline{a+n}\times\&c.\times\overline{a+pn}}+\frac{O}{\overline{a+2n}\times\&c.\times\overline{a+p+1n}}+\&c.$$

ubi numeratores M, N, O, &c. constituunt Seriem quamlibet terminorum, quorum differentiæ, vel primæ, vel secundæ, vel aliæ quædam dantur; vel quod perinde est, qui constituunt lineam quamvis transversam in dato quovis triangulo Arithmetico; Denominatores autem constituunt Seriem in *Prop.* I. exhibitam.

Solutio. Per X designetur primus factorum a, $a+n$, $a+2n$, &c. in denominatore ejusdem termini, ut sint X & z iidem ac in *Lemm*: præmissis, adeoque designetur terminus quilibet Seriei per $\dfrac{X}{z\times\overline{z+n}\times\&c.\times\overline{z+p-n}}$

Per *Lem.* 2, vel per *Lem.* 4. (prout magis commodum

 videatur

videatur vel differentias, vel generatores trianguli Arithmetici adhibere,) resolvatur X in Multinomium $A + B \times z + Cz \times \overline{z+n} + Dz \times \overline{z+n} \times \overline{z+2n} + \&c.$ Hoc pacto (terminis multinomii ad denominatorem $z \times \overline{z+n} \times \&c. \times \overline{z+p-n}$, applicatis) terminus quilibet Seriei revocabitur ad formulam $\frac{A}{z \times \overline{z+n} \times \&c. \times \overline{z+p-1n}}$

$$+ \frac{B}{\overline{z+n} \times \&c. \times \overline{z+p-1n}} + \frac{C}{\overline{z+2n} \times \&c \times \overline{z+p-1n}}$$

$+ \&c.$

Unde (per Scholium 4 *Prop.* I.) aggregatum totius Seriei, à termino $\frac{X}{z \times \overline{z+n} \times \&c. \times \overline{z+p-1n}}$ inclusive in infinitum continuatæ, est

$$\frac{A}{\overline{p-1} \times n \times z \times \overline{z+n} \times \&c. \times \overline{z+p-2n}}$$

$$+ \frac{B}{\overline{p-2} \times n \times \overline{z+n} \times \&c. \times \overline{z+p-2n}}$$

$$+ \frac{C}{\overline{p-3} \times n \times \overline{z+2n} \times \&c. \times \overline{z+p-2n}} + \&c.$$

re si dematur hoc aggregatum ab ejusdem aggregati valore quando $z = a$, residuum erit summa omnium terminorum ante terminum $\frac{X}{z + \&c.}$, hoc est, tot terminorum quot sunt unitates in $\frac{z-a}{n}$. *Q. E. I.*

Ex. 1. Sit primum exemplum in Serie $\frac{5}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13}$

+

$+ \frac{41}{5\cdot7\cdot9\cdot11\cdot13\cdot15} + \frac{131}{7\cdot9\cdot11\cdot13\cdot15\cdot17}$
$+ \frac{275}{9\cdot11\cdot13\cdot15\cdot17\cdot19} + \frac{473}{11\cdot13\cdot15\cdot17\cdot19\cdot21}$
$+$ &c. Sunt hic $a = 3$, $n = 2$, $t = 5$, $M = 5$, & capiendo differentias numeratorum inveniuntur $b = 36$, $c = 54$, $d = 0 = e =$ &c. Hinc in Lemmate secundo sunt $A = 5 + 36 \times \frac{-3}{2} + 54 \times \frac{-3}{2} \times \frac{-5}{4} = \frac{209}{4}$, $B = \frac{1}{2} \times 36 + 54 \times \frac{-5}{2} = \frac{-99}{2}$, $C = \frac{1}{2} \times \frac{1}{4} \times 54 = \frac{27}{4}$, $D = 0 = E =$ &c. Summa itaque totius Seriei est $\frac{209}{4\times5\times2\times3\cdot5\cdot7\cdot9\cdot11} + \frac{-99}{2\times4\times2\times5\cdot7\cdot9\cdot11}$ $+ \frac{27}{4\times3\times2\times7\cdot9\cdot11} = \frac{283}{80\times3\cdot5\cdot7\cdot9\cdot11}$, atque summa terminorum numero $\frac{z-3}{2}$ $(= \frac{z-a}{n})$ est

$$\frac{283}{80\times3\cdot5\cdot7\cdot9\cdot11} - \frac{209}{40\times z.\overline{z+2}.\overline{z+4}.\overline{z+6}.\overline{z+8}}$$
$$+ \frac{99}{16\times\overline{z+2}\times\overline{z+4}.\overline{z+6}.\overline{z+8}} - \frac{27}{24\times\overline{z+4}.\overline{z+6}.\overline{z+8}}$$

Quærantur *v. g.* octo termini; tum existente $\frac{z-3}{2} = 8$ fit $z = 19$, quo valore in formulâ adhibito, prodit summa $\frac{155891}{2\cdot3\cdot3\cdot3\cdot3\cdot5\cdot5\cdot5\cdot7\cdot11\cdot19\cdot23}$.

Iidem Numeratores occupant lineam tertiam transversam in Triangulo Arithmetico

$$\begin{array}{l} 54\cdot54\cdot54\cdot54\cdot\ 54\cdot\ 54\cdot \text{\&c.} \\ -18\cdot36\cdot90\cdot144\cdot198\cdot\text{\&c.} \\ \quad 5\cdot41\cdot131\cdot275\cdot\text{\&c.} \end{array}$$

Unde

Unde in formula *Lem.* 4. ſunt generatores $q = 5$, $r = -18$; $s = 54$, $t = 0 = \&c.$ & prodeunt coefficientes $A = 5 - 18 \times \frac{-3}{2} + 54 \times \frac{-3}{2} \times \frac{-3+2}{4} = \frac{209}{4}$, $B = \frac{1}{2} \times \overline{-18 + 54 \times \frac{-3}{2}} = \frac{-99}{2}$, $C = \frac{1}{2} \times \frac{1}{4} \times 54 = \frac{27}{4}$, $D = 0 = E = \&c.$ iidem ac ſupra.

Ex. 2. Sit Series $\frac{4}{1.2.3.4.5.6.7.8.9.10.11} + \frac{69}{2.3.\&c.12} + \frac{530}{3.4.\&c.13} + \frac{2676}{4.5.\&c.14} + \frac{10350}{5.6.\&c.15} + \&c.$ Ubi ſunt $a = 1$, $n = 1$, $p = 11$, atque Numeratores conſtituant Seriem in *Corol.* 20. *Lem.* 4. exhibitam. Applicando itaque valorem X in *Corol.* illo ad denominatorem $z \times \overline{z+1} \times \&c. \times \overline{z+10}$, fit Seriei propoſitæ Terminus

$$\frac{-1}{1.2.3.4.5.6 \times \overline{z+6}.\overline{z+7}.\overline{z+8}.\overline{z+9}.\overline{z+10}}$$
$$+ \frac{23}{1.2.3.4.5.6.7 \times \overline{z+7}.\overline{z+8}.\overline{z+9}.\overline{z+10}}$$
$$- \frac{72}{1.2.3.4.5.6.7.8 \times \overline{z+8}.\overline{z+9}.\overline{z+10}}$$
$$+ \frac{54}{1.2.3.4.5.6.7.8.9 \times \overline{z+9} \times \overline{z+10}}.$$ Adeoque per hanc *Prop.* ſumma Seriei à termino illo in infinitum continuatæ eſt

$$\frac{-1}{4 \times 1.2.3.4.5.6 \times \overline{z+6}.\overline{z+7}.\overline{z+8}.\overline{z+9}}$$

+

$$+\frac{23}{3\times 1.2.3.4.5.6.7\times\overline{z+7}.\overline{z+8}.\overline{z+9}}$$

$$-\frac{72}{2\times 1.2.3.4.5.6.7.8\times\overline{z+8}.\overline{z+9}}$$

$$+\frac{54}{1\times 1.2.3.4.5.6.7.8.9\times\overline{z+9}}.$$

Itaque pro z sumpto 1, fit summa totius Seriei $\frac{305}{12\times 1.2.3.4.5.6.7.8.9.10}$. Et in genere summa terminorum numero $\frac{z-1}{1}$, est $\frac{305}{12\times 1.2.3.4.5.6.7.8.9.10}$

$$+\frac{1}{4\times 1.2.3.4.5.6\times\overline{z+6}.\overline{z+7}.\overline{z+8}.\overline{z+9}}$$

$$-\frac{23}{3\times 1.2.3.4.5.6.7\times\overline{z+7}.\overline{z+8}.\overline{z+9}}$$

$$+\frac{72}{2\times 1.2.3.4.5.6.7.8\times\overline{z+8}\times\overline{z+9}}$$

$$-\frac{54}{1\times 1.2.3.4.5.6.7.8.9\times\overline{z+9}}.$$

Scholium 1. In computandis summis hujusmodi Serierum, calculus plerumque levior est adhibitis generatoribus trianguli Arithmetici, quam si adhibeantur differentiæ. Libet itaque hac occasione ostendere quomodo ex datis differentiis inveniri possunt generatores Trianguli Arithmetici.

Sunto itaque ω primus Seriei terminus, a differentia ultima data, b prima differentiarum penultimarum, c prima antepenultimarum, & sic porrò d, e, *&c.* atque sint t, u, x, y, *&c* generatores quæsiti Trianguli Arithmetici, cujus lineam transversam ordine p occupet Series

Hhhhh pro-

proposita. Tum (quod ex contemplatione Trianguli Arithmetici facile constat) sunt

$$a = t$$

$$b = \frac{p-1}{1} t + u$$

$$c = \frac{p-1}{1} \times \frac{p-2}{2} t + \frac{p-2}{1} u + x$$

$$d = \frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3} t + \frac{p-2}{1} \times \frac{p-3}{2} u$$

$$+ \frac{p-3}{1} x + y \text{ \&c.}$$

Unde colliguntur generatorum valores

$$t = a$$

$$u = b - \frac{p-1}{1} t$$

$$x = c - \frac{p-1}{1} \times \frac{p-2}{2} t - \frac{p-2}{1} u$$

$$y = d - \frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3} t - \frac{p-2}{1} \times \frac{p-3}{2} u$$

$$- \frac{p-3}{1} x \text{ \&c.}$$

Ultimus autem generator æqualis est Seriei termino primo ω.

2. D[nus] de *Monsoury Abbas Orbacensis* mihi amicissimus, & ruri vicinus, postquam cum eo hæc communicaveram, aliam invenit hujus Problematis Solutionem, cujus formulam ob ejus miram simplicitatem hic referre juvat. Itaque in Serie numeratorum sint ω terminus primus, b prima differentiarum primarum, c prima secundarum, d prima tertiarum, & sic porrò; atque sit termini primi Denominator $z \times \overline{z+n} \times \text{\&c.} \times \overline{z+p-1\,n}$; Tum summa

summa totius Seriei in infinitum continuatæ exhibebitur per formulam

$$\frac{\omega}{n \times \overline{p-1} \times z \times \overline{z+n} \times \&c. \times \overline{z+p-2n}} + \frac{b}{n^2 \times \overline{p-1} \times \overline{p-2} \times z \times \overline{z+n} \times \&c. \times \overline{z+p-2n}} + \frac{c}{n^3 \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \overline{z+2n} \times \&c. \times \overline{z+p-2n}}$$

$+ \&c.$

Sit exemplum in Serie $\frac{5}{3 \cdot 5 \cdot \&c. \, 13} + \frac{41}{5 \cdot 7 \cdot \&c. \, 15} + \frac{131}{7 \cdot 9 \cdot \&c. \, 17} + \frac{275}{9 \cdot 11 \cdot \&c. \, 19} + \&c.$ cujus summam jam exhibuimus. In hoc casu sunt $\omega = 5$, $b = 36$, $c = 54$, $d = 0 = e = \&c.$ Unde per formulam summa Seriei integræ fit $\frac{5}{2 \cdot 5 \times 3 \cdot 5 \ldots 11} + \frac{36}{4 \cdot 5 \cdot 4 \times 5 \ldots 11} + \frac{54}{8 \cdot 5 \cdot 4 \cdot 3 \times 7 \ldots 11} = \frac{283}{80 \times 3 \cdot 5 \ldots 11}$, ut per formulam nostram exhibetur. Si quæratur summa ejusdem Seriei incipientis à termino decimo $\frac{2273}{21 \ldots 31}$, in eo casu $\omega = 2273$, $b = 522$, $c = 54$, & summa esset

$$\frac{2273}{2 \cdot 5 \times 21 \ldots 29} + \frac{522}{4 \cdot 5 \cdot 4 \times 23 \ldots 29} + \frac{54}{8 \cdot 5 \cdot 4 \cdot 3 \times 25 \ldots 29}$$

Hæc formula est commodissima, & summam exhibet nullo ferè negotio, quoties quæritur summa Seriei integræ, & differentiæ non sunt nimis multæ. Sed ubi plures sunt differentiæ, & quæritur non Series integra, sed termini tantùm initiales aliquammulti, formulæ nostræ sunt commodiores.

3. Quando

3. Quando Serierum termini formantur tantùm per Multiplicationem, nec afficiuntur divisoribus variabilibus, summæ semper exhiberi possunt per Methodum in *Prop.* I. traditam, sint licet formulæ quantumlibet compositæ. Nam possunt semper revocari ad terminos in formâ quam postulat Propositio illa. Sic si differentiæ ipsorum z & x sint m & n, & designetur terminus Seriei per $z\,x$; hic terminus revocabitur ad formam $\overline{a - n}\,z + \frac{n}{m}\,z \times \overline{z + m}$; cujus Integrale datur *per Prop.* I; nempe quoniam $dx = n$, & $dz = m$, est $dx = dz \times \frac{n}{m}$; unde regrediendo ad integralia fit $x = \frac{n}{m}\,z + a$ (adjecto invariabili a, ut habeatur ratio relationis inter z & x in Seriei termino primo,) quod sic scribi potest $\overline{a - n} + \frac{n}{m} \times \overline{z + m}$, ut deinde in z ductum induat formam requisitam. Et ad eundem modum procedere licet in aliis casibus ejusmodi. Sed ubi formulæ oblatæ divisoribus afficiuntur, eædem ac in Calculo integrali, ut vocant, difficultates occurrunt, eâdem industriâ superandæ. Nec tamen semper superari possunt. Nam præterquam quod vix certò sciri possit quæ debeat relatio intercedere inter Numeratorem fractionis & Denominatorem, ut formula oblata ad Integrale revocari possit; sæpe etiam difficillimum est explorare an adsit jam talis relatio in formulâ istâ, aut si desit, an introduci possit. Quicquid ego in hâc materiâ potissimum inveni, continetur in tribus sequentibus propositionibus.

Prop. III. *Prob.*

Crescentibus, z, u, y, x, *&c.* per differentias datas n, m, l, o, *&c.* invenire valorem numeratoris integri

tegri N, ut existente Denominatore $z . \overline{z+n} . \&c. \overline{z+pn} \times u . \overline{u+m} . \&c. \overline{u+qm} \times y . \overline{y+l} . \&c. \overline{y+rl} \times x . \overline{x+o}$ &c. $\overline{x+so}$. &c. Fractio ad Integrale revocari possit.

Solutio. Fiat $N = \overline{z+pn} \times \overline{u+qm} \times \overline{y+rl} \times \overline{x+so} \times \&c. - zuyx\ \&c.$ atque Integrale erit fractio, cujus Denominator $z . \overline{z+n} . \&c. \overline{z+\overline{p-1}\,n} \times u . \overline{u+m} . \&c. \overline{u+\overline{q-1}\,m} \times y . \overline{y+l} . \&c. \overline{y-\overline{x-1}\,l} \times x . \overline{x+o} . \&c. \overline{x+\overline{s-1}\,o} \times \&c.$ existente 1 Numeratore.

Differentia enim hujus fractionis est fractio cujus numerator est ipsius N valor exhibitus, & denominator idem est ac denominator propositus, ut fieri debuit.

Ex. 1. Sit denominator propositus $z \times \overline{z+2} \times u \times \overline{u+3}$. In hoc casu sunt $n=2$, $m=3$, $p=1$, $q=1$; adeoque est $N = \overline{z+2} \times \overline{u+3} - zu = 3z+2u+6$, & per $\frac{3z+2u+6}{z . \overline{z+2} \times u . \overline{u+3}}$ representatur terminus Seriei summabilis, cujus nempe in infinitum continuatæ summa exhibetur per $\frac{1}{zu}$. Sint verbi gratiâ, ipsorum z & u primus valor communis 1, atque Series summabilis erit $\frac{11}{1 . 3 \times 1 . 4} + \frac{23}{3 . 5 \times 4 . 7} + \frac{35}{5 . 7 \times 7 . 10} + \&c.$, quippe cujus totius summa est 1. Per p designetur ordo termini cujusvis in hâc Serie, erit $p = \frac{z-1+2}{2} = \frac{u-1+3}{3}$, adeoque $z = 2p-1$, & $u = 3p-2$; quibus valoribus pro z & u scriptis, designabitur terminus per formulam $\frac{12p-1}{\overline{2p-1} \times \overline{2p+1} \times \overline{3p-2} \times \overline{3p+1}}$. Summa autem terminorum omnium ante terminum illum, hoc est terminorum initialium numero $\frac{z-1}{2} = p-1$, est

$1 - \frac{1}{zu}, = \frac{zu-1}{zu}$, hoc est $\frac{6pp-7p+1}{\overline{2p-1}\times\overline{3p-2}}$. Quare pro p scripto $p+1$, erit $\frac{p\times\overline{6p+5}}{\overline{2p+1}\times\overline{3p+1}}$ aggregatum tot terminorum initialium quot sunt unitates in p.

Ex. 2. Iisdem manentibus z, u, n, m, sit denominator $z.\overline{z+2}.\overline{z+4}\times u.\overline{u+3}$. Tum per formulam numerator erit $\overline{z+4}\times\overline{u+3}-zu=3z+4u+12$, & summa Seriei exhibebitur per formulam $\frac{1}{z.\overline{z+2}\times u}$. Sit ipsorum z & u primus valor communis 1, & hinc elicietur Series $\frac{19}{1.3.5\times1.4}+\frac{37}{3.5.7\times4.7}+\frac{55}{5.7.9\times7.10}$ + &c $=\frac{1}{3}$.

Scholium. In Seriebus jam expositis eadem ubique est differentia inter factores continuos ejusdem cujusvis termini, ac inter factores homologos terminorum continuorum. In sequentibus exempla quædam sunt Serierum, quarum summæ in terminis numero finitis exhiberi possunt, quamvis ea regula non observetur.

Prop. IV. *Prob.*

Crescente z per differentias datas qn, invenire numeratorem integrum N, ut ad Integrale revocari possit fractio, cujus Denominator sit ex certo numero p terminorum z, $z+n$, $z+2n$, &c. Arithmeticè proportionalium in invicem ductorum. Debet autem esse q numerus integer minor quam factorum numerus p.

Solutio. Erit $N=\overline{z+\overline{p-1}\,n}\times\overline{z+\overline{p-2}\,n}\times$ &c. $\times\overline{z+\overline{p-q}\,n}-z\times\overline{z+n}\times$ &c. $\times\overline{z+\overline{q-1}\,n}$, Integrale

tegrale exiftente $\frac{1}{z \times \overline{z+n} \times \&c. \times \overline{z+p-q-1\,n}}$ Demonftratur ad modum propofitionis præcedentis.

Sumptis ad libitum n, p, q, & primo valore z, hinc oriuntur infinitæ Series fummabiles, cujufmodi funt Series tres fequentes.

$$A = \frac{5}{1.2.3.4} + \frac{9}{3.4.5.6} + \frac{13}{5.6.7.8} + \frac{17}{7.8.9.10} \&c.$$

$$B = \frac{1}{1.2.3.4.5} + \frac{4}{4.5.6.7.8} + \frac{9}{7.8.9.10.11} + \frac{16}{10.11.12.13.14} + \&c.$$

$$C = \frac{1}{1.2.3.4.5} + \frac{14}{5.6.7.8.9} + \frac{55}{9.10.11.12.13} + \frac{140}{13.14.15.16.17} + \&c.$$

Has Series jampridem communicavi cum primariis quibufdam Geometris, à quibus minimè contemni videntur. Sic ad me fcribit periciffimus Geometra D *Nicolaus Bernoulli* in epiftolâ datâ 25 *Julii* 1716. " Vous " me ferez un extreme plaifir, *Monfieur*, de me com-" muniquer la Solution de voftre probleme, *Etant donnée* " *une fuitte des Fractions dont les Numerateurs foient des* " *nombres figurés quelconque,* & *dont les Denominateurs* " *foient formés du produit d'un nombre egal de Facteurs* " *qui foient en Progreffion Arithmetique, trouver la fom-*" *me*; & principalement comment vous avez trouvé " ces deux formules $\frac{p}{24 \times \overline{4p+1}}$, $\frac{p.\overline{p+1}}{12 \times \overline{3p+1} \times \overline{3p+2}}$.

Hæ formulæ fpectant ad Series C & B, defignante p numerum terminorum, quorum fumma requiritur. Sic etiam ad me fcribit D. *Taylor* in epiftola datâ 22 *Aug.* 1716. " Ut & quâ ratione incidifti in fummationem " Serierum à te exhibitarum, præfertim loquor de

Serie

" Serie $\frac{1}{1.2.3.4.5} + \frac{4}{4.5.6.7.8} + \frac{9}{7.8.9.10.11} +$ &c.
" quæ videtur esse altioris indaginis.

Sed ut ad exempla jam redeamus. In Serie *A* sunt $p = 4$, $q = 2$, $n = 1$, primo valore z existente 1. Est itaque $\overline{z+3} \times \overline{z+2} - z \times \overline{z+1} = 2 \times \overline{2z+3}$ formula, unde (rejecto dato numero 2) derivantur numeratores 5, 9, 13, 17, &c. Formula etiam summæ est $\frac{1}{z \times \overline{z+1}}$. Quare habitâ ratione numeri 2, quem ex numeratoribus rejecimus, summa totius Seriei, à termino in quo est z in infinitum continuatæ, exhibetur per formulam $\frac{1}{2z \times \overline{z+1}}$; adeoque summa Seriei integræ est $\frac{1}{2 \times 1 \times 2} = \frac{1}{4}$.

In Serie *B* sunt $n = 1$, $p = 5$, $q = 3$, primo valore z existente 1. Est itaque $N = \overline{z+4} \times \overline{z+3} \times \overline{z+2} - z \times \overline{z+1} \times \overline{z+2} = 6 \times \overline{z+2}|^2$. Ipsius autem $\overline{z+2}$ valores continui sunt 3, 6, 9, &c. qui quoniam omnes sunt divisibiles per 3, ponendo $z + 2 = 3x$, fit $N = 6 \times \overline{3x}|^2 = 6 \times 9x^2 = 54x^2$, ipsius x valoribus continuis existentibus 1, 2, 3, &c. Rejecto itaque numero dato 54, hinc prodeunt numeratores 1, 2^2, 3^2, &c. hoc est 1, 4, 9, &c. Formula etiam Integralis est $\frac{1}{z \times \overline{z+1}}$; quare habitâ ratione numeri 54 quem ex numeratoribus rejecimus, summa Seriei à termino in quo est z in infinitum continuatæ est $\frac{1}{54z \times \overline{z+1}}$. Unde summa Seriei integræ est $\frac{1}{108}$.

In Serie denique *C* sunt $n = 1$, $p = 5$ $q = 4$, & primus valor $z = 1$. Unde fit $N = \overline{z+4} \times \overline{z+3} \times \overline{z+2} \times \overline{z+1} - z \times \overline{z+1} \times \overline{z+2} \times \overline{z+3} = 4 \times \overline{z+1}$

×

$\overline{z+2}\times\overline{z+3}$. Valores autem N per hanc formulam prodeuntes semper possunt dividi per $4\times 2\times 3\times 4=96$. Ergò hoc divisore rejecto prodeunt numeratores 1, 14, 55, 140, *&c.* Et formula Summæ, habitâ ratione numeri 96, est $\frac{1}{96z}$. Adeoque Summa Seriei integræ est $\frac{1}{96}$.

Scholium 1. Per Propositiones has duas novissimas nullo negotio inveniri possunt Series quot libuerit summabiles. Et vicissim oblatâ Serie hujus speciei, si summari potest, ejus summa plerumque revocatur ad alterutram ex his Propositionibus. In examine tamen solertiâ est opus. Optime autem procedit si termini Seriei oblatæ revocentur ad formulam *Prop.* III. Sic *e. gr.* propositâ Serie $\frac{7}{3\cdot 5\cdot 7\cdot 9\cdot 11}+\frac{11}{7\cdot 9\cdot 11\cdot 13\cdot 15}+\frac{15}{11\cdot 13\cdot 15\cdot 17\cdot 19}+$ *&c.* Denominatores sic scribi possunt $3\cdot 7\cdot 11\times 5\cdot 9$, $7\cdot 11\cdot 15\times 9\cdot 13$, $11\cdot 15\cdot 19\times 13\cdot 17\cdot$ *&c.*

Unde juxta *Prop.* III. fit $n=4$, $m=4$, $p=2$, $q=1$, primus valor $z=3$, primus valor $u=5$. Hinc formula Numeratoris invenitur $4\times\overline{z+2u+8}$, Est autem $z+2u+8$ semper divisibile per 3; quare rejectis divisoribus datis 4 & 3, per hanc formulam prodeunt Numeratores 7, 11, 15, *&c.* iidem ac Numeratores in Serie proposita, quæ proinde summabitur per illam propositionem.

2. Cùm Series illas *A*, *B*, *C*, communicaveram cum D. *Taylor*, rescripsit se earum summas invenisse primam quidem *A* & tertiam *C*, eas revocando ad casus simplices Methodi Incrementorum, tertiam *C. e g.* revocavit ad hanc formam $\frac{1}{24}\times\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\frac{1}{13\cdot 17}$ *&c.*, ut habeatur summa per præcepta tradita in *Scholio Prop* 1.

 In

In Serie autem secundâ B, cùm hoc non æquè successit, sequenti usus est Analysi, quam, ipsius venia jam impetratâ, ob ejus eximiam elegantiam huc transferre non piget. " Seriei istius terminus [in Stylo ejus] exhibetur per formulam $\frac{\overline{z+2}\times z}{27z\times\overline{z+1}\times\underset{,}{z}\times\overline{\underset{,}{z}+1}}$; pro $z+3$ in denominatore scripto $\underset{,}{z}$, quoniam est $\underset{\cdot}{z}=3$. Pone $\frac{B}{27C}$ æquale esse Integrali quæsito, hoc est $\frac{B}{C}$ esse Integrale ipsius $\frac{\overline{z+2}\times z}{z\cdot\overline{z+1}\times\underset{,}{z}\cdot\overline{\underset{,}{z}+1}}$, seposito divisore dato 27. Ipsius autem $\frac{B}{C}$ incrementum est $\frac{\underset{\cdot}{B}C-B\underset{\cdot}{C}}{C\underset{,}{C}}$. Debet ergo $\frac{\underset{\cdot}{B}C-B\underset{\cdot}{C}}{C\underset{,}{C}}$ idem esse ac $\frac{\overline{z+2}\times z}{z\cdot\overline{z+1}\times\underset{,}{z}\cdot\overline{\underset{,}{z}+1}}$. Comparando denominatores invenitur $C=z\times\overline{z+1}$. Hinc itaque sumendo incrementa fit $\underset{\cdot}{C}=2z\underset{\cdot}{z}+\underset{\cdot}{z}^2+\underset{\cdot}{z}$ ($=2\underset{\cdot}{z}z+4\underset{\cdot}{z}$, quoniam est $\underset{\cdot}{z}=3$,) His valoribus in locum C & $\underset{\cdot}{C}$ substitutis prodit $\underset{\cdot}{B}C-B\underset{\cdot}{C}=\overline{zz+z}\,\underset{\cdot}{B}-2\underset{\cdot}{z}\times\overline{z+2}\,B$, quod debet esse idem ac $\overline{z+2}\times z$. Sit $B=a+v$, existente a ipsius B parte invariabili, & v parte variabili. Tum sumendo incrementa fit $\underset{\cdot}{B}=\underset{\cdot}{v}$. Unde ad invenienda a & v habetur æquatio $\overline{zz+z}\,\underset{\cdot}{v}-2\underset{\cdot}{z}\times\overline{z+2}\times\overline{a+v}=\overline{z+2}\times z$, quæ sic scribi potest $\overline{zz+z}\,\underset{\cdot}{v}-2\underset{\cdot}{z}\times\overline{z+2}\,v=\underset{\cdot}{z}\times\overline{z+2}\times\overline{1+2a}$, vel etiam $C\underset{\cdot}{v}-\underset{\cdot}{C}v=\underset{\cdot}{z}\times\overline{z+2}\times\overline{1+2a}$. Pone $1+2a=0$ (unde fit $a=\frac{-1}{2}$,) & fit $C\underset{\cdot}{v}-\underset{\cdot}{C}v=0$;

" ubi

" ubi fieri poteſt $v = 0$, (quoniam æquationis termini
" ſinguli afficiuntur vel ab v, vel ab $\dot{v}$) Hinc ergò fit $B =$
" $a = \frac{-1}{2}$, adeoque $\frac{B}{C} = \frac{-1}{2z\times\overline{z+1}}$. Unde habitâ ra-
" tione diviſoris 27, Integrale quæſitum fit $\frac{-1}{54\times z\times\overline{z+1}}$.
" Sed & comparando æquationem $C\dot{v} - \dot{C}v = 0$ cum
" formulâ generali $\frac{B\dot{C} - \dot{B}C}{CC} = 0$, inde etiam conclude-
" re licet eſſe $\frac{v}{C}$ = quantitati datæ, (quoniam ipſius
" incrementum eſt 0.) Unde pro n ſumpto quovis
" numero dato, fit $v = nC$, atque $B = -\frac{1}{2} + nC$.
" Quo pacto Integrale quæſitum fit $\frac{B}{C} = \frac{-\frac{1}{2} + nC}{C} = \frac{-1}{2C}$
" $+ n$, quod ab Integrali prius invento differt quan-
" titate datâ n. Hoc inde fit, quòd, ut in quadraturâ
" Curvarum Area inventa augeri poteſt vel minui areâ
" datâ, ſic in Methodo incrementorum Integrale inven-
" tum augeri poteſt vel minui quantitate datâ. Per
" Integrale autem primum, ubi deeſt n, exhibetur
" ſumma Seriei in infinitum continuatæ.

Prop. V.

Creſcente z per unitates, & exiſtentibus a, b, c, &c. numeris datis integris, quorum nullæ inter ſe æquantur; invenire Integrale ipſius $\frac{1}{z\times\overline{z+a}\times\overline{z+b}\times\overline{z+c}\times\&c.}$

Solutio. Ducendo tam numeratorem quam denominatorem fractionis in terminos $z+1$, $z+2$, &c. $z+a+1$, $z+a+2$, &c. $z+b+1$, $z+b+2$, &c. $z+c+1$, $z+c+2$, &c. in denominatore deficientes, revocetur Denominator ad formulam $z\times\overline{z+1}$

×

$\times\overline{z+2}\times\&c.$ denominatoris in *Prop.* I. *Schol. n.* 3. Deinde revocetur Numerator ad formam $A + Bz + Cz \times \overline{z+1} + Dz \times \overline{z+1} \times \overline{z+2} + \&c.$ Tum applicando terminos ad Denominatorem novum $z \times \overline{z+1} \times \overline{z+2} \times \&c.$ revocetur fractio ad hanc formam

$$\frac{A}{z \times \overline{z+1} \times \&c.} + \frac{B}{\overline{z+1} \times \overline{z+2} \times \&c.} + \frac{C}{\overline{z+2} \times \overline{z+3} \times \&c.} + \frac{D}{\overline{z+3} \times \overline{z+4} \times \&c.}\ \&c.$$

Unde denique quæratur Integrale per *Schol. Prop.* I. *n.* 3.

Ratio Solutionis per se satis est manifesta.

Scholium 1. Hujus Solutionis tota difficultas latet in revocatione numeratoris ad formam requisitam, quod tamen quomodo sit faciendum uno exemplo patebit. Proponatur itaque factum $\overline{z+2} \times \overline{z+3} \times \overline{z+7}$, quod ad formam propositam sit revocandum. Terminos itaque evolvo gradatim ut sequitur. Factorem primum $z+2$ sic scribo $2+z$, cujus terminum primum 2 duco in $3+z$, unde fit $6+2z$: Terminum secundum z duco in $\overline{2+z+1}$ $(=z+3)$ unde fit $2z + z \times \overline{z+1}$. Dein facta in unam summam colligendo, fit $\overline{z+2} \times \overline{z+3} = \begin{matrix} 6+2z \\ +2z \end{matrix} + z \times \overline{z+1} = 6 + 4z + z \times \overline{z+1}$. Superest ut hoc ducatur in $z+7$. Itaque terminum primum 6 duco in $7+z$ $(=z+7)$ unde fit $42+6z$; terminum secundum $4z$ duco in $\overline{6+z+1}$ $(=z+7)$ unde fit $24z + 4z \times \overline{z+1}$; terminum tertium $z \times \overline{z+1}$ duco in $\overline{5+z+2}$ $(=z+7,)$ unde fit $5z \times \overline{z+1} + z \times \overline{z+1} \times \overline{z+2}$. Factis itaque in unum collectis ut prius, fit $\overline{z+2} \times \overline{z+3} \times \overline{z+4} = 42 + 30z + 9z \times \overline{z+1} + z \times \overline{z+1} \times \overline{z+2}$. Et ad eundem modum procedere licet in aliis casibus.

2. Sit

2. Sit autem exemplum Propositionis in fractione $\frac{1}{z\times\overline{z+2}\times\overline{z+5}}$. Restituendo factores $z+1$, $z+3$, $z+4$ in Denominatore deficientes, fractio fit $\frac{\overline{z+1}\times\overline{z+3}\times\overline{z+4}}{z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}$. Revocandus itaque est Numerator $\overline{z+1}\times\overline{z+3}\times\overline{z+4}$ ad formam requisitam. Itaque per methodum jam traditam fit primo $\overline{z+1}\times\overline{z+3}=1\times\overline{3+z}+z\times\overline{2+z+1}$ $=3+\overline{z+2z}+z\times\overline{z+1}=3+3z+z\times\overline{z+1}$. Deinde $\overline{z+1}\times\overline{z+3}\times\overline{z+4}=3\times\overline{4+z}+3z\times\overline{3+z+1}+z\times\overline{z+1}\times\overline{2+z+2}=12+3z+9z$ $+3z\times\overline{z+1}+2z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$ $=12+12z+5z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$. Applicando hoc factum ad Denominatorem $z\times\overline{z+1}\times$ &c. $\times\overline{z+5}$ fractio tandem revocatur ad hanc formam

$$\frac{12}{z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}$$
$$+\frac{12}{\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}$$
$$+\frac{5}{\overline{z+2}\times\overline{z+3}\times\overline{z+4}\times\overline{z+5}}+\frac{1}{\overline{z+3}\times\overline{z+4}\times\overline{z+5}}$$

Cujus denique Integrale est $\frac{-12}{5z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}}$

$$+\frac{-12}{4\cdot\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{-5}{3\cdot\overline{z+2}\times\overline{z+3}\times\overline{z+4}}$$
$$\frac{-1}{2\cdot\overline{z+3}\times\overline{z+4}}.$$

3. Quando duo tantùm sunt factores z & $z+a$, exhibebitur etiam Integrale per formulam $\frac{1}{2}-\frac{1-a}{2z\times\overline{z+1}}$

$$-\frac{\overline{1-a}\times\overline{2-a}}{3z\times\overline{z+1}\times\overline{z+2}}-\frac{\overline{1-a}\times\overline{2-a}\times\overline{3-a}}{4z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}}\ \text{\&c.}$$

Seriem nempe continuando donec abrumpatur per eva-

 nescentiam

nescentiam terminorum. Si Factores duo sint z & $z-a$ exhibebitur Integrale per formulam $\frac{-1}{z-1} - \frac{-1+a}{2.\overline{z-1}.\overline{z-2}} - \frac{-1+a\times-2+a}{3.\overline{z-1}.\overline{z-2}.\overline{z-3}} - \&c.$ Potest idem Integrale exprimi utroque modo, prout fractionis oblatæ factor vel minor vel major sumatur pro z.

4. Si primus valor z sit $a+1$, migrabit formula posterior in hanc $\frac{-1}{a}\times\overline{\frac{1}{1}\times\frac{1}{2}\times\frac{1}{3}+\&c.}$ usque $\frac{1}{a}$ inclusivè, quâ, cum signo contrario, exhibetur summa Seriei $\frac{1}{1\times\overline{1+a}}+\frac{1}{2\times\overline{2+a}}+\frac{1}{3\times\overline{3+a}}+\&c.$ in infinitum continuatæ. Sit *e. gr.* $a=1$, atque Series erit $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\&c.=\frac{1}{1}\times\frac{1}{1}=1$. Si $a=2$, erit Series $\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\&c.=\frac{1}{2}\times\overline{\frac{1}{1}+\frac{1}{2}}=\frac{3}{4}$; Si $a=3$, Series erit $\frac{1}{1\times4}+\frac{1}{2\times5}+\frac{1}{3\times6}+\frac{1}{4\times7}$ &c. $=\frac{1}{3}\times\overline{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}}=\frac{11}{18}$.

5. Ex eâdem Serie $\frac{1}{1\times\overline{1+a}}+\frac{1}{2\times\overline{2+a}}+\frac{1}{3\times\overline{3+a}}+\&c.$ pro diverso valore a oriuntur Series plures formâ satis elegantes, quarum nonnullas Lectori ob oculos sistere, credo, ingratum non erit.

Si pro a sumantur successivè numeri pares, 2, 4, 6, 8, &c. Series erunt

Si $a=2$) $\frac{1}{1\times\overline{1+2}}+\frac{1}{2\times\overline{2+2}}+\frac{1}{3\times\overline{3+2}}+\frac{1}{4\times\overline{4+2}}+\&c.$

4) $\frac{1}{1\times\overline{1+4}}+\frac{1}{2\times\overline{2+4}}+\frac{1}{3\times\overline{3+4}}+\frac{1}{4\times\overline{4+4}}+\&c.$

6) $\frac{1}{1\times\overline{1+6}}+\frac{1}{2\times\overline{2+6}}+\frac{1}{3\times\overline{3+6}}+\frac{1}{4\times\overline{4+6}}+\&c.$

8) $\frac{1}{1\times\overline{1+8}}+\frac{1}{2\times\overline{2+8}}+\frac{1}{3\times\overline{3+8}}+\frac{1}{4\times\overline{4+8}}+\&c.$

Vel

$$\text{Vel } \frac{1}{4-1} + \frac{1}{9-1} + \frac{1}{16-1} + \frac{1}{25-1} + \&c.$$

$$\frac{1}{9-4} + \frac{1}{16-1} + \frac{1}{25-4} + \frac{1}{36-4} + \&c.$$

$$\frac{1}{16-9} + \frac{1}{25-9} + \frac{1}{36-9} + \frac{1}{49-9} + \&c.$$

$$\frac{1}{25-16} + \frac{1}{36-16} + \frac{1}{49-16} + \frac{1}{64-16} + \&c.$$

$$\text{Vel } \frac{1}{4-1} + \frac{1}{9-1} + \frac{1}{16-1} + \frac{1}{25-1} + \&c.$$

$$\frac{1}{4+1} + \frac{1}{9+3} + \frac{1}{16+5} + \frac{1}{25+7} + \&c.$$

$$\frac{1}{4+3} + \frac{1}{9+7} + \frac{1}{16+11} + \frac{1}{25+15} + \&c.$$

$$\frac{1}{4+5} + \frac{1}{9+11} + \frac{1}{16+17} + \frac{1}{25+23} + \&c.$$

Si pro a ſumantur ſucceſſive numeri impares 1, 3, 5, 7, &c. Series erunt

$$=1)\ \frac{1}{1 \times \overline{1+1}} + \frac{1}{2 \times \overline{2+1}} + \frac{1}{3 \times \overline{3+1}} + \frac{1}{4 \times \overline{4+1}} + \&c.$$

$$3)\ \frac{1}{1 \times \overline{1+3}} + \frac{1}{2 \times \overline{2+3}} + \frac{1}{3 \times \overline{3+3}} + \frac{1}{4 \times \overline{4+3}} + \&c.$$

$$5)\ \frac{1}{1 \times \overline{1+5}} + \frac{1}{2 \times \overline{2+5}} + \frac{1}{3 \times \overline{3+5}} + \frac{1}{4 \times \overline{4+5}} + \&c.$$

$$7)\ \frac{1}{1 \times \overline{1+7}} + \frac{1}{2 \times \overline{2+7}} + \frac{1}{3 \times \overline{3+7}} + \frac{1}{4 \times \overline{4+7}} + \&c.$$

$$\text{Vel } \frac{1}{2} \times \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \&c.$$

$$\frac{1}{2} \times \frac{1}{3-1} + \frac{1}{6-1} + \frac{1}{10-1} + \frac{1}{15-1} + \&c.$$

$$\frac{1}{2} \times \frac{1}{6-3} + \frac{1}{10-3} + \frac{1}{15-3} + \frac{1}{21-3} + \&c.$$

$$\frac{1}{2} \times \frac{1}{10-6} + \frac{1}{15-6} + \frac{1}{21-6} + \frac{1}{28-6} + \&c.$$

$$\text{Vel } \frac{1}{2} \times \frac{1}{1+0} + \frac{1}{3+0} + \frac{1}{6+0} + \frac{1}{10+0} + \&c.$$

$$\frac{1}{2} \times \frac{1}{1+1} + \frac{1}{3+2} + \frac{1}{6+3} + \frac{1}{10+4} + \&c.$$

$$\frac{1}{2} \times \frac{1}{1+2} + \frac{1}{3+4} + \frac{1}{6+6} + \frac{1}{10+8} + \&c.$$

$$\frac{1}{2} \times \frac{1}{1+3} + \frac{1}{3+6} + \frac{1}{6+9} + \frac{1}{10+12} + \&c.$$

6 Ante

6. Ante aliquot annos D. *Jac. Bernoulli* Geometra insignis invenit summam Seriei cujusslibet, cujus Numeratores constituunt Seriem æqualium, Denominatores verò constituunt, vel Seriem quadratorum dato aliquo quadrato *Q* minutorum, vel Seriem Triangulorum, dato aliquo Triangulo *T* minutorum. Hæc invenit ille observando quod hujusmodi Series oriantur ex ablatione Seriei Harmonicè proportionalium truncatæ ab eâdem Serie integrâ; nempe ita ut numerus terminorum deficientium in Serie truncata, sit, vel duplus lateris dati quadrati *Q*, vel duplus unitate auctus lateris dati Trianguli *T*. Idem etiam observavit frustrà quæri summam Seriei reciprocæ Quadratorum. Hoc idem etiam verum est de reciprocis Cuborum, vel aliarum quarumlibet dignitatum numerorum in progressione Arithmeticâ. Ratio est, quòd nulla intercedit differentia inter factores denominatorum, quod ad hujusmodi summationes semper requiri constat ex Methodo sumendi differentias in *Scholio Prop.* I. jam explicatâ. Nam si per formulam aliquam exhiberi posset summa quæsita, differentia istius formulæ exhiberet terminos Seriei propositæ: sed in tali differentiâ denominator semper afficitur per factores ab invicem diversos, quod quoniam in Seriebus prædictis non obtinet, summæ Serierum hujusmodi in terminis finitis haberi nequeunt. Ad eundem ferè modum, argumento petito à *Prop.* III. & IV. demonstrari potest summas Serierum exhiberi non posse in terminis numero finitis, quarum Numeratores constituunt Seriem æqualium. Denominatores vero constant ex certo numero terminorum in progressione Arithmeticâ, maximo factore cujusvis termini minore existente quàm factor minimus in termino proxime insequenti, cujusmodi est Series $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&c.$

7. Jam liceret regulas nonnullas tradere quas pro casibus quibusdam singularibus concinnavi; sed hæc

nos

nos longius abducerent. Sufficiat itaque quæ generaliora sunt explicasse, & simul monuisse, ad novæ hujusce Serierum infinitarum doctrinæ provectionem nihil magis facere, quam si excogitentur formulæ generaliores summarum, ex quarum differentiis, per regulas supra traditas computatis, deinde conficiantur Canones quantitatum summabilium; ita ferè ut jam factum est in Calculo Integrali, *h. e.* in Stylo *Newtoniano*, in Methodo Fluxionum.

8. Restituendo factores in Denominatore deficientes potuisset præsens Problema revocari ad *Propositionem* II. Sed & in terminis generalioribus proponi potest, nempe pro Numeratore sumptâ quâvis Formulâ, cujus differentia aliqua datur. Sub eâ tamen conditione ut dimensiones Denominatoris ad minimum binario superent Dimensiones Numeratoris; aliàs enim summa Seriei in terminis numero finitis haberi nequit. Sit hujus rei exemplum in Serie $\frac{1}{1.3.5.7} + \frac{4}{2.4.6.8} + \frac{9}{3.5.7.9} + \frac{16}{4.6.8.10} +$ *&c.* ubi Numeratores sunt numerorum naturalium quadrata. Applicando tum Numeratores tum Denominatores ad numeros naturales, Series revocatur ad formam simpliciorem $\frac{1}{3.5.7} + \frac{2}{4.6.8} + \frac{3}{5.7.9} + \frac{4}{6.8.10} +$ *&c.* Per p designatis numeris naturalibus 1, 2, 3, 4, *&c.* terminus Seriei designabitur per formulam $\frac{p}{\overline{p+2} \times \overline{p+4} \times \overline{p+6}}$; vel per formulam $\frac{z-2}{z \times \overline{z+2} \times \overline{z+4}}$, nempe pro $p+2$ scripto z. Quoniam progrediendo de termino in terminum augetur z per unitates, restituendi sunt factores in denominatore deficientes $z+1$, $z+3$, & hoc pacto revocatur terminus Seriei ad formulam $\frac{\overline{z-2} \times \overline{z+1} \times \overline{z+3}}{z \times \overline{z+1} \times \overline{z+2} \times \overline{z+3} \times \overline{z+4}}$. Per methodum in hâc Propositione jam explicatam re-

 vocatur

vocatur numerator ad formam $-6-6z-z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$. Unde habita ratione denominatoris Terminus revocatur ad formam $\frac{-6}{z\times\overline{z+1}\times\&c\times\overline{z+4}}+\frac{-6}{\overline{z+1}\times\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{-1}{\overline{z+2}\times\overline{z+3}\times\overline{z+4}}+\frac{1}{\overline{z+3}\times\overline{z+4}}$. Adeoque sumendo Integrale fit $\frac{6}{4z\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}}+\frac{6}{3\times\overline{z+1}\times\overline{z+2}\times\overline{z+3}}+\frac{1}{2\times\overline{z+2}\times\overline{z+3}}+\frac{-1}{\overline{z+3}}$; quo, sub signo contrario, exhibetur summa Seriei in infinitum continuatæ, incipientis à termino $\frac{z-2}{z\times\overline{z+2}\times\overline{z+4}}$. Summa itaque Seriei integræ incipientis à termino $\frac{1}{3\cdot5\cdot7}$ est $\frac{31}{240}$.

Si per *Prop.* II. procedere esset animus, ex formulâ $z-2\times\overline{z+1}\times\overline{z+3}$ collectis numeratoribus primis 24, 70, 144, 252, sumendo eorum differentias haberentur $46=b$, $28=c$, $6=d$, $e=0=$ &c. existente $M=24$; unde per *Lem.* 2. prodiret formula $-6-6z-z\times\overline{z+1}+z\times\overline{z+1}\times\overline{z+2}$, quâ designatur Terminus, eadem ac supra; atque pergendo per *Prop.* II. haberetur summa.

Prop. VI. *Prob.*

Invenire summam quotlibet terminorum Seriei Fractionum, quarum Numeratores & Denominatores constituunt lineas duas quasvis transversas in Triangulo Arithmetico *Paschalii*; nempe cujus generatores sunt unitates.

Solutio. Per n designetur Ordo Seriei Numeratorum in Triangulo Arithmetico, & sit p differentia inter ordinem Numeratorum & Denominatorum, & per q designetur numerus terminorum quorum summa requiritur

quiritur. Tum si Denominatores sint plurium dimensionum quàm sunt Numeratores, Summa exhibebitur per formulam primam sequentem; si dimensiones Numeratorum plures sint quàm dimensiones Denominatorum, Summa exhibebitur per formulam secundam.

Formula I.

$$\frac{n+p-1}{p-1} - \frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}\,.\,\&c.\,\overline{n+p-1}}{\overline{p-1}\times\overline{n+q}\,.\,\overline{n+q+1}\,.\,\&c.\,\overline{n+q+p-2}}$$

Formula II.

$$-\frac{n-p-1}{p+1} + \frac{\overline{q+n-1}\,.\,\overline{q+n-2}\,.\,\&c.\,\overline{q+n-p-1}}{\overline{p+1}\times\overline{n-1}\,.\,\overline{n-2}\,.\,\&c.\,\overline{n-p}}$$

Ex. 1. Inveniendum sit aggregatum sex primorum terminorum Seriei $\frac{1}{1}+\frac{4}{7}+\frac{10}{28}+\frac{20}{84}+\frac{35}{210}+\frac{56}{462}+$ &c. ubi Numeratores constituunt lineam quartam, Denominatores constituunt lineam septimam in Triangulo Arithmetico Sunt itaque $n=4$, $p=3$, $q=6$; & quoniam dimensiones Denominatorum superant dimensiones Numeratorum, dabitur summa per Formulam primam; nempe $\frac{4+3-1}{3-1} - \frac{4\,.\,5\,.\,6}{\overline{3-1}\times\overline{4+6}\times\overline{4+7}}$ sive $3-\frac{6}{11}=2\frac{5}{11}$.

Ex. 2. Quæratur summa sex primorum terminorum Seriei $\frac{1}{1}+\frac{7}{4}+\frac{28}{10}+\frac{84}{20}+\frac{210}{35}+\frac{462}{56}+$ &c. cujus termini sunt terminorum Seriei prioris reciproci. Sunt itaque $n=7$, $p=3$, $q=6$, adeoque per formulam secundam summa fit $-\frac{3}{4}+\frac{12\,.\,11\,.\,10\,.\,9}{4\times 6\,.\,5\,.\,4}=24$.

Scholium 1. Formulas in hac propositione exhibitas ante biennium communicavi cum Viris celeberrimis *Moivreo* & *Bernoulliis*. Facile autem derivari possunt ex præceptis in *Prop.* I. traditis. Sit exemplum in Serie priori $\frac{1}{1}+\frac{4}{7}+\frac{10}{28}+$ &c. Per p designato loco

Ter-

Termini in Serie hâc, exhibetur Terminus per formulam $\frac{4 \cdot 5 \cdot 6}{\overline{p+3} \cdot \overline{p+4} \cdot \overline{p+5}}$; Unde regrediendo ad Integrale, summa Seriei incipientis à termino illo exhibetur per formulam $\frac{4 \cdot 5 \cdot 6}{2 \times \overline{p+3} \times \overline{p+4}}$; adeoque pro p sumpto 1, Series integra fit $\frac{4 \cdot 5 \cdot 6}{2 \cdot 4 \cdot 5} = 3$, atque summa primorum sex terminorum fit $3 - \frac{4 \cdot 5 \cdot 6}{2 \cdot 10 \cdot 11}$, omninò ut per formulam jam exhibetur.

2. In formulâ primâ summa Seriei in infinitum continuatæ est $\frac{n+p-1}{p-1}$, evanescente jam parte alterâ formulæ. Sed in casu formulæ secundæ summa hæc est infinitum quid, cujus species, respectu numeri infiniti q, exhibetur per formulæ partem alteram, quæ in hoc casu fit $\frac{q^{p+1}}{\overline{p+1} \times \overline{n-1} \cdot \overline{n-2} \cdot \&c. \overline{n-p}}$.

3. De hujusmodi Seriebus in epistolâ datâ mense *Maio* 1716, sic ad me scripsit Vir. Ill. *D. Leibnitius*, quem magno Scientiarum damno nobis nuper ereptum lugemus. " Il me semble qu'autrefois j'ay aussi sommé " quelques Series ou suittes comme $\frac{1}{1} + \frac{2}{4} + \frac{3}{10} + \frac{4}{20}$ " $+ \frac{5}{35} + \frac{6}{56} +$ *&c.* Le terme de cette suitte exprimé " Analytiquement est $\frac{x}{x \cdot \overline{x+1} \cdot \overline{x+2} \times \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3}}$ " $= \frac{1 \cdot 2 \cdot 3}{\overline{x+1} \cdot \overline{x+2}} = \frac{6}{xx+3x+2}$. On demande donc " la somme d'une suitte donneé, dont un terme soit " $\frac{ll}{xx+3lx+2ll}$ ou x signifie les nombres naturales " 1, 2, 3, 4, *&c.* & l signifie l'Unité, ou la difference " des x. Supposons que le terme de la suitte som-

" matrice

“ matrice demandée soit $\frac{fx}{mx+nl} = \frac{\odot}{\text{☽}}$. Or Diff. $\frac{\odot}{\text{☽}} =$
“ $-\frac{\odot}{\text{☽}} + \frac{\odot + d\odot}{\text{☽} + d\text{☽}} = \frac{\text{☽}\, d\odot - \odot\, d\text{☽}}{\text{☽}\text{☽} + \text{☽}\, d\text{☽}}$: sed $d\odot = f dx$,
“ & $d\text{☽} = m\,dx = ml$; donc la Difference de $\frac{\odot}{\text{☽}}$ est =
“ $\frac{nfll}{mmxx + 2mnlx + nnll + mmlx + mnll}$. Maintenant il faut faire

$$\frac{nfll}{mmxx + 2mnlx + nnll + mmlx + mnll} = \frac{mfll}{mmxx + 3mmlx + 2mmll}$$

“ c'est a dire, il faut identifier ces deux formules, ou la
“ donnée est Multipliée per $\frac{nf}{mm}$: donc égalant les
“ termes respectifs, puisque les xx conviennent, on
“ aura par les x, $2n+m = 3m$, c'est adire il y aura
“ $m = n$, & par les absolus on aura $nn + mn = 2mm$,
“ ce qui donne encore $m = n$; donc l'identification
“ reussit, & nous pouvons faire $n = m = l = 1$, &
“ $f = 1$ (car f demeure arbitraire) & le terme de la
“ suitte sommatrice sera $\frac{x}{x+1}$, car diff. $\frac{x}{x+1}$ donne
“ $-\frac{x}{x+1} + \frac{x+1}{x+2} = \frac{1}{xx+3x+2}$, & par consequente
“ $\frac{6x}{x+1}$ donne la somme des $\frac{x}{x.\overline{x+1}.\overline{x+2} \times \frac{1}{1}.\frac{1}{2}.\frac{1}{3}}$

“ $3,\ 4,\ \frac{9}{2},\ \frac{24}{5},\ 5,\ \frac{36}{7}$, *&c. Series summatrix, cujus ter-*
“ *minus* $\frac{6x}{x+1}$.

“ $\frac{1}{1} + \frac{2}{4} + \frac{3}{10} + \frac{4}{20} + \frac{5}{35} +$ *&c. Series summanda, cu-*
“ *jus terminus* $\frac{x}{x.\overline{x+1}.\overline{x+2} \times \frac{1}{1} \times \frac{1}{2}.\frac{1}{3}}$ Et pour
“ s'en servir aux sommations, les 5 termes, *par Ex.* de

 “ la

" la ſuitte donnée feront $\frac{36}{7} - 3 = \frac{15}{7}$. Et generalle-
" ment la ſomme des termes juſqu'a quelque terme
" $\frac{x}{x\,.\,\overline{x+1}\;\overline{x+2}\times\frac{1}{1}\,.\,\frac{1}{2}\,.\,\frac{1}{3}}$ excluſivement, ſera $\frac{6x}{x+1}$
" -3 : Et pour la ſomme de la ſuitte entiere a l'infi-
" nie, x devient infini, & $\frac{6x}{x+1} = 6$: donc la ſomme
" de toute la ſuitte eſt $6 - 3 = 3$, comme vous
" l'avez trouvé.

" Cette methode eſt le calcul des differences ap-
" pliqué aux Nombres ; & il faut vous avouer qu'a-
" vant que de l'appliquer aux Figures, & même avant
" que d'avoir été Geometre, Je le prattiquai en quel-
" que façon dans les nombres ; ayant trouvé encore
" jeune garçon que les ſuittes dont les Numerateurs
" fuſſent des Unites, & dont les Denominateurs fuſſent
" les Nombres figurés, comme Triangulaires Pyrami-
" daux *&c.* etoient les differences 1^{eres}, 2es, 3emes, *&c.*
" multipliées par les conſtantes de la ſuitte $\frac{1}{1} + \frac{1}{2} + \frac{1}{3}$
" $+ \frac{1}{4} +$ *&c.* & par conſequent ſommables. Mais
" quand je devins un peu Geometre & Analyſte, Je
" vis qu'il y avoit moyen de venir a bout de telles
" ſommations par une Methode generalle, autant qu'il
" etoit poſſible ; & que le calcul des differences eſtoit
" encore plus commode dans la Geometrie que dans
" les Nombres, puis qu'il y a plus d'evanouiſſements,
" & que les differences repondent aux Tangentes, les
" ſommes aux Quadratures. Cette methode generalle
" de chercher la ſuitte ſommatrice de la ſuitte donnée,
" quand elle eſt poſſible, reuſſit toujours, quand le terme
" de la ſuitte donnée exprimé Analytiquement n'a
" point la quantité variable enveloppé dans une racine,
" ny entrant dans l'expoſant ; & alors, on peut tou-
" jours

" jours determiner la ſuitte ſommatrice, ou prouver
" qu'il eſt impoſſible d'en trouver. Et la choſe reuſſit
" méme bien ſouvent, lors méme que la variable en-
" tre dans l'Expoſant. Mais comme il y a quelque-
" fois des Quadratures particulieres de quelques por-
" tions d'une Figure, dont ou ne ſçauroit donner la
" Quadrature generalle ou la Figure quadratrice; de
" méme on peut trouver quelqueſois la ſomme de
" toute la ſuitte, ou d'un certaine partie, quoy qu'on
" ne puiſſe pas trouver la ſomme de chaque partie; &
" alors il faut avoir recours a des Methodes particulieres,
" dont on n'eſt pas toujours le maiſtre, noſtre Analyſe
" n'eſtant pas encore portée a ſa perfection.

Prop. VII. *Prob.*

Invenire ſummam Seriei cujus Numeratores conſtituunt lineam quamlibet erectam in Triangulo Arithmetico *Paſchalii*, Denominatores vero conſtituunt lineam quamlibet tranſverſam.

Solutio. Deſignetur ordo lineæ erectæ per p, ordo lineæ tranſverſæ per q, & ſit m aggregatum tot terminorum primorum in lineâ erectâ ordinis $p+q-1$ quot ſunt unitates in $q-1$, atque ſumma quæſita erit $\overline{2^{p+q-2}-m} \times \frac{1 \,.\, 2 \,.\, 3 \,.\, \&c. \,\overline{q-1}}{p \,.\, \overline{p+1} \,.\, \&c. \,\overline{p+q-2}}$.

Ex. 1. Proponatur Series $\frac{1}{1}+\frac{5}{4}+\frac{10}{10}+\frac{10}{20}+\frac{5}{35}+\frac{1}{56}$ Ubi Numeratores conſtituunt lineam ſextam erectam, Denominatores occupant lineam quartam tranſverſam. In hoc itaque caſu ſunt $p=6$, $q=4$, $p+q-1=9$, $q-1=3$, adeoque $m=1+8+28=37$ *i e.* tribus terminis primis lineæ nonæ erectæ. Unde fit ſumma quæſita $\overline{2^8-37} \times \frac{1 \,.\, 2 \,.\, 3}{6 \,.\, 7 \,.\, 8}=\frac{219}{56}$.

Ex. 2. Conſtituant Numeratores lineam centeſimam erectam, & ſint Denominatores Numeri Trigonales, qui occupant lineam tertiam tranſverſam. Tum erunt

p

$p = 100$, $q = 3$, $m = 102$ atque adeo ſumma quæſita ſit $\overline{2^{101} - 102} \times \frac{1 \cdot 2}{100 \cdot 101}$.

Cor. Si $q = 2$, formula ſit $\frac{2^p - 1}{p}$, quâ exhibetur aggregatum primi termini, unâ cum ſemiſſe ſecundi, triente tertii, quadrante quarti, & ſic porrò, lineæ cujuſvis erectæ ordinis p Trianguli Arithmetici *Paſchalli*. Sic *v. gr.* eſt $\frac{1}{1} + \frac{5}{2} + \frac{10}{3} + \frac{10}{4} + \frac{5}{5} + \frac{1}{6} = \frac{2^6 - 1}{6} = 10\frac{1}{2}$.

Prop. VIII. *Prob.*

Invenire ſummam ejuſdem Seriei, quando terminorum ſigna ſunt alternatim + & —.

Solutio. Summa quæſita exhibetur per formulam ſimpliciſſimam $\frac{q-1}{p+q-2}$.

Ex. Invenienda ſit ſumma Seriei $\frac{1}{1} - \frac{6}{9} + \frac{15}{45} - \frac{20}{165} + \frac{15}{495} - \frac{6}{1287} + \frac{1}{3003}$, ubi Numeratores conſtituunt lineam ſeptimam erectam, Denominatores conſtituunt nonam tranſverſam. In formulâ itaque pro p & q ſcriptis 7 & 9, ſit ſumma $\frac{8}{14}$.

Manente eâdem Serie Numeratorum (nempe lineâ ſeptimâ erectâ), ſi pro Serie Denominatorum ſumantur ſucceſſivè lineæ tranſverſæ 2^{da}, 3^{tia}, 4^{ta}, *&c.* Summæ erunt $\frac{1}{7}$, $\frac{2}{8}$, $\frac{3}{9}$, $\frac{4}{10}$, $\frac{5}{11}$, *&c.* quæ ſic poſſunt ſcribi, $\frac{1}{7}$, $\frac{7}{28}$, $\frac{28}{84}$, $\frac{84}{210}$, $\frac{210}{462}$, *&c.* ubi tam Numeratores, quàm Denominatores excerpuntur ex lineâ tranſverſâ ordinis ſeptimi. Idem eveniret ſi loco ſeptimæ, Numeratores conſtituiſſent aliam quamlibet lineam erectam ordinis p; Summæ quippe orirentur ex applicatione terminorum

lineæ

lineæ transversæ ejusdem ordinis p ad terminos proximè sequentes in eadem lineâ.

Propositiones hæ duæ novissimæ potius elegantes sunt quàm utiles; quare Formularum nostrarum demonstrationem Lectoris solertia investigandam relinquimus, ad Propositionem ultimam jam properantes, quæ tertiam continet Serierum speciem, ob usum multiplicem satis insignem.

Lemma 5.

Sit Series quævis $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4},$ *&c.* cujus terminorum Denominatores constituunt progressionem quamlibet Geometricam $h, h^2, h^3, h^4,$ *&c.* Sint etiam Numeratorum primus A $(=M)$, prima differentiarum primarum B, prima secundarum C, prima tertiarum D, quartarum E, & sic porrò; & sint $\frac{\alpha}{h}, \frac{\beta}{h^2}, \frac{\gamma}{h^3}, \frac{\delta}{h^4},$ *&c.* respectivè, aggregata, Unius, Duorum, Trium, Quatuor, vel plurium terminorum Seriei $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3},$ *&c.* atque sint Numeratorum primus a $(=\alpha)$ prima differentiarnm primarum b, prima secundarum c, prima tertiarum d, & sic porrò: & sit $h-1=q$. Tum ipsorum $a, b, c, d,$ *&c.* valores erunt.

$$a = A = \alpha = M$$
$$b = hA + B$$
$$c = q\,hA + hB + C$$
$$d = q^2 hA + qhB + hC + D$$

& sic porrò.

Demonstratio.

Satis constat esse $a = \alpha = A = M$.

Termini $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4},$ *&c.* Numeratoribus $M, N, O, P,$

&c. expressis per A, B, C, D, *&c.* transformantur in terminos $\frac{A}{h}$, $\frac{A+B}{h^2}$, $\frac{A+2B+C}{h^3}$, $\frac{A+3B+3C+D}{h^4}$ &c. Unde colligendo summas terminorum, inveniuntur Numeratores α, β, γ, δ, &c. nempe

$$\alpha = A$$
$$\beta = \overline{h+1}\,A + B$$
$$\gamma = \overline{h^2+h+1}\,A + \overline{h+2}\,B + C$$
$$\delta = \overline{h^3+h^2+h+1}\,A + \overline{h^2+2h+3}\,B + \overline{h+3}\,C + D$$

&c.

Unde sumendo differentias fiunt

$$b = hA + B$$
$$c = qhA + hB + C$$
$$d = qqhA + qhB + hC + D$$

& sic porrò, ut in Propositione exhibentur.

Cor. 1. Si Numeratorum M, N, O, P, *&c.* differentia vel prima, vel secunda, vel alia quædam detur, terminis omnibus post primos aliquot in Serie A, B, C, D, *&c.* evanescentibus, Differentiæ b, c, d, &c. tandem incurrent in Progressionem Geometricam in ratione 1 ad q. Exempli gratiâ, si detur Numeratorum M, N, O, P *&c* differentia prima B, erunt c, d, *&c.* in ratione continuâ Geometricâ 1 ad q; ut constat per ipsorum valores $qhA + hB$, $qqhA + qhB$, *&c.* existentibus $C = 0 = D = $ *&c.*

Cor. 2. Ordo autem primæ differentiarum B, C, D, *&c.* quæ hoc modo evanescunt, idem est ac ordo differentiæ vel b, vel c, *&c.* unde incipit Progressio illa Geometrica. Sic si $B = 0 = C = $ *&c.* erunt b, c, d, *&c.* in Progressione Geometricâ; si $C = 0 = D = $ *&c.* erunt c, d, *&c.* in Progressione Geometricâ. Et sic porrò.

Lemma 6.

Iisdem positis sit r terminus unde incipit Progressio Geometrica in Serie differentiarum b, c, d, *&c.* & per

$p + 1$

$p+1$ designetur ordo Termini in Serie $\frac{\alpha}{b}, \frac{\beta}{b^2}, \frac{\gamma}{b^3}, \frac{\delta}{b^4}$, &c. Tum Terminus ille designabitur per fractionem cujus Denominatore existente b^{p+1} Numerator est

$$\overline{a+bp+cp\times\frac{p-1}{2}+dp\times\frac{p-1}{2}\times\frac{p-2}{3}+\&c.+\frac{r}{q^n}}$$
$$\times\, b^p - 1 - qp - q^2p\times\frac{p-1}{2} - q^3p\times\frac{p-1}{2}\times\frac{p-2}{3} - \&c.$$

nempe per n designato ordine differentiæ evanescentis in Serie B, C, D, &c. ut & Numero terminorum $a+bp$, &c. item terminorum $-1-qp$, &c.

Demonstratio. Per *Lemma* 1. Termini istius Numerator exhibetur per formulam

$$a+bp+cp.\frac{p-1}{2}+dp\times\frac{p-1}{2}.\frac{p-2}{3}+\&c.(p+1$$

subeunte vices x in Lemmate isto)

Ergò si sit, *ex. gr.* $n=2$, per *Lemm.* 5. *Cor.* 2. erunt c, d, &c. in ratione continuâ 1 ad q. Numerator itaque in hoc casu est

$$a+bp+cp\times\frac{p-1}{2}+cqp\times\frac{p-1}{2}\times\frac{p-2}{3}+cq^2p$$
$$\times\frac{p-1}{2}\times\frac{p-2}{3}\times\frac{p-3}{4}+\&c.$$ Sed si termini $cp\times\frac{p-1}{2}$ $+cqp\times\frac{p-1}{2}\times\frac{p-2}{3}+\&c.$ ducantur in $\frac{q^2}{c}$, & producti addantur termini $1+qp$, prodibit Series quâ exprimitur binomii $1+q$ dignitas $\overline{1+q}|^p=b^p$. Ergo productum illud æquale est b^p-1-qp; adeoque termini $cp\times\frac{p-1}{2}+cqp\times\frac{p-1}{2}\times\frac{p-2}{3}+\&c.=\frac{c}{q^2}$ $\times\overline{b^p-1-qp}$. Quo pacto Numerator fit $a+bp$ $+\frac{c}{q^2}\times\overline{b^p-1-qp}$, existentibus duobus terminis $a+bp$, ut & duobus $-1-qp$, juxta sensum Propositionis, quoniam $n=2$. Atque eadem est demonstratio in aliis casibus. De Denominatore verò per se satis constat.

Prop. IX. *Prob.*

Invenire summam quotlibet terminorum Seriei cujusvis $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$, &c. cujus terminorum Denominatores constituunt progressionem quamlibet Geometricam h, h^2, h^3, h^4, &c. Numeratores autem sunt quantitates differentiâ aliquâ constanti gaudentes.

Solutio Sunto Numeratorum M, N, O, P, &c. primus A, prima differentiarum primarum B, prima secundarum C, prima tertiarum D, & sic porrò; & sit ipsorum A, B, C, D, &c. numerus n, atque $h - 1 = q$, Tum fiat $a = A (= M)$ $b = h A + B$, $c = q h A + h B + C$, $d = q^2 h A + q h B + h C + D$, &c. ut sint tot termini a, b, c, d, &c, quot sunt unitates in $n + 1$. Terminorum istorum ultimus dicatur r, atque per $p + 1$ designetur numerus terminorum $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$, &c. quorum summa requiritur; Dico summam illam exhiberi per fractionem, cujus Denominatore existente h^{p+1}, Numerator est

$$\frac{a + bp + cp \times \frac{p-1}{2} + dp \times \frac{p-1}{2} \times \frac{p-2}{3} + \&c. + \frac{r}{q^n}}{\times h^p - 1 - qp - q^2 p \times \frac{p-1}{2} - q^3 p \times \frac{p-1}{2} \times \frac{p-2}{3} - \&c.}$$

$$\overline{- q^{n-1} p \times \frac{p-1}{2} \times \&c.}$$

Demonstratio. Nam (per *Lem.* 6.) per hanc formulam repræsentatur terminus ordine $p + 1$ Seriei $\frac{\alpha}{h}, \frac{\beta}{h^2}, \frac{\gamma}{h^3}, \frac{\delta}{h^4}$, &c. qui terminus (per constructionem *Lemmatis* 5.) æqualis est aggregato terminorum numero $p + 1$ Seriei propositæ $\frac{M}{h}, \frac{N}{h^2}, \frac{O}{h^3}, \frac{P}{h^4}$. *Q. E. D.*

Ex. 1. Invenienda ſit ſumma novem terminorum Seriei $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{8}$, $\frac{4}{16}$, &c. Sunt in hoc caſu $h = 2$, q $(= h - 1) = 1$, $p + 1 = 9$, $p = 8$, $A = 1$, $B = 1$, $C = 0, = D = \&c.$ adeoque $n = 2$, (quoniam ſunt duo A, B,) Hinc fit $a (= A) = 1$, $b (= hA + B = 2 \times 1 + 1)$ $= 3$, $c (= qhA + hB + C = 2 \times 1 + 2 \times 1 + 0)$ $= 4 = r$, Adeoque per formulam fit ſumma quæſita

$$\frac{1 + 3 \times 8 + \frac{4}{1^2} \times \overline{2^3 - 1 - 1 \times 8}}{2^9} = \frac{1013}{512}.$$

Ex. 2. Quæratur ſumma ſex terminorum Seriei $1 \times 3 + 3 \times 3^2 + 6 \times 3^3 + 10 \times 3^4 + 15 \times 3^5 + 21 \times 3^6 + \&c.$ In hoc caſu ſunt $h = \frac{1}{3}$, $q = \frac{-2}{3}$, $p + 1 = 6$, $p = 5$, $A = 1$, $B = 2$, $C = 1$, $D = 0 = E = \&c.$ adeoque $n = 3$, atque $a = 1$, $b = \frac{1}{3} + 2 = \frac{7}{3}$, $c = \frac{-2}{9} + \frac{2}{3} + 1 = \frac{13}{9}$, $d = \frac{4}{27} - \frac{4}{9} + \frac{1}{3} = \frac{1}{27} = r$. Unde ſumma quæſita fit $= 19956$. ſive

$$\frac{1 + \frac{7}{3} \times 5 + \frac{13}{9} \times 5 \times \frac{4}{2} + \frac{-1}{8} \times \overline{\frac{1}{3^5} - 1 + \frac{2}{3} \times 5 - \frac{4}{9} \times 5 \times \frac{4}{2}}}{\overline{\frac{1}{3}}|^{6}}$$

Cor. 1. Ejuſdem Seriei, à termino primo $\frac{M}{h}$ in infinitum continuatæ, ſumma exhibetur per formulam ſimpliciſſimam $\frac{A}{h - 1} + \frac{B}{\overline{h - 1}|^2} + \frac{C}{\overline{h - 1}|^3} + \frac{D}{\overline{h - 1}|^4}$ &c.

Cor. 2. Si $h = 2$, Seriei totius in infinitum continuatæ ſumma habetur ſolâ additione terminorum A, B, C, D, &c. Et hæc ſumma eadem eſt ac ſumma lineæ erectæ reſpondentis termino primo A, in Triangulo Arithmetico, cujus lineam tranſverſam occupant Nume-

ratores M, N, O, P, *&c.* Quod facile conſtat ex contemplatione Trianguli. Si itaque fuerint M, N O, *&c.* Numeri figurati cujuſvis ordinis n, ſumma Seriei $\frac{M}{2} + \frac{N}{4} + \frac{O}{8} + \frac{P}{16} +$ *&c.* æqualis erit Numeri binarii dignitati $\overline{2}|^{n-1}$. Sic Series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ *&c.* $= 2^{1-1} = 1$, ut vulgò notum; Series $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} +$ *&c.* $= 2^{n-1} = 2$; Series $\frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} +$ *&c.* $= 2^{3-1} = 2^2 = 4$, & ſic porrò.

Scholium. Celeb. D. *Jac. Bernoulli*, in Tractatu ſuo de Seriebus infinitis, ſolvit illud Problema. " Invenire " ſummam Seriei infinitæ Fractionum quarum Denomi- " natores creſcunt in Progreſſione quacunque Geome- " tricâ, Numeratores verò progrediuntur vel juxta Nu- " meros naturales, 1, 2, 3, 4, *&c.* vel Trigonales 1, " 3, 6, 10, *&c.* vel Pyramidales 1, 4, 10, 20, *&c.* " aut juxta Quadratos 1, 4, 9, 16, *&c.* aut Cubos 1, " 8, 27, 64, *&c.* eorumve multiplices." Ipſius ſolutionem conſulat Lector. Aliam verò, & quidem multo generaliorem invenit D. *Nic. Bernoulli* illius Nepos, eamque (poſtquam ei hæc miſeram, ſed ſine demonſtratione) mecum communicare dignatus eſt, in epiſtolâ datâ 18° *Septembris* 1715, miris quidem inventis referciſſimâ, qualibus me crebro dignatur vir Doctiſſimus. De hoc vero Problemate ſic ſcribit. " Pour la ſomme " d'un nombre determiné n de termes de la ſuitte de " voſtre Theoreme 7. [*Corollarium* primum eſt hujus Propoſitionis] j'ay trouvé cette formule $\frac{1}{m^n} \times$

$$\times \overline{\frac{m-1}{m-1}a + \frac{A-n}{m-1}b + \frac{B-n.\frac{n-1}{2}}{m-1}c + \frac{C-n.\frac{n-1}{2}.\frac{n-2}{3}}{m-1}d}$$

" + *&c.* ou les Lettres A, B, C, *&c.* marquent

" les

" les Coefficients des termes immediatement prece-
" dents. Et en mettant dans cette formule $p + r$
" pour n, h^m pour m, & en multipliant tout encore
" par e^{m-1}, on a la solution de vostre *Prob.*
" IX^{me}". Et me monuit Vir peritissimus hanc suam formulam generalem in nostram particularem (*Cor.* 1. hujus propositionis) migrare quando $n = \infty$; quippe tum evanescunt $1, n, n \cdot \frac{n-1}{2}, n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, &c respectu ipsorum m^n, A, B, C, &c. adeo ut Series in eo casu sit $\frac{1}{m-1} a + \frac{A}{m-1} b + \frac{B}{m-1} c +$ &c. quæ omnino coincidit cum nostrâ $\frac{a}{m-1} + \frac{b}{\overline{m-1}|^2} + \frac{c}{\overline{m-1}|^3} +$ &c.

Adhuc aliam hujus Problematis solutionem, & quidem ab hisce admodum diversam, invenit D. *Taylor* ope Methodi suæ Incrementorum. Viri doctissimi rogatu, ad eum miseram formulam meam secundam pro solutione Problematis II^{di}, item formulas alias spectantes ad Propositiones tertiam, quartam & quintam, sed sine demonstrationibus: quippe non dubitabam quin Vir acutissimus, atque ipse Methodi istius Incrementorum Inventor, hisce, vel saltem paribus inveniendis par esset. Rescripsit se harum solutiones invenisse, & simul alia quædam communicavit ad hujus methodi profectum multum facientia, quæ jam nostro hortatu inductus hisce subjungere dignatur.

Appendix

APPENDIX

Quâ methodo diversâ eadem materia tractatur:
Auctore Brook Taylor, *LL. D. R. S. Secr.*

HOrtatu Viri Clariss. cui nos innumeris officiis devinctissimos esse libenter fatemur, sequentes jam Propositiones exhibemus, quas quidem in aliam occasionem reservandas esse decrevissemus, ni æquum visum fuisset parendum esse imperio amici qui, dum Propositiones quasdam præcedentes suas olim nobis investigandas proposuit, earum inveniendarum occasionem dedit.

Definitiones.

1. Quantitatis cujusvis variabilis valorem præsentem designo literâ simpliciter scriptâ, ut x; valores præcedentes distinguo lineolis eidem literæ ex parte superiori positis, sequentes lineolis ex parte inferiori scriptis. Ut vi hujus Definitionis sint $\overset{\prime\prime}{x}$, $\overset{\prime}{x}$, x, $\underset{\prime}{x}$, $\underset{\prime\prime}{x}$, ejusdem variabilis valores quinque continui, existente x valore præsenti, $\overset{\prime}{x}$ proximè præterito, $\overset{\prime\prime}{x}$ secundò præterito; $\underset{\prime}{x}$ proximè, atque $\underset{\prime\prime}{x}$ secundò futuro. Et sic de aliis. Ad eundem modum sunt interpretandæ lineolæ quæ incrementis apponuntur. Sic sunt $\overset{\prime\prime}{\underset{..}{x}}$, $\overset{\prime}{\underset{..}{x}}$, $\underset{..}{x}$, $\underset{\prime..}{x}$, $\underset{\prime\prime..}{x}$, ipsius $\underset{..}{x}$ valores quinque continui; ut sit $\overset{\prime\prime}{\underset{..}{x}}$ incrementum secun-

secundum ipsius $\underset{\cdot}{x}''$, sit $\underset{\cdot\cdot}{x}'$ incrementum secundum ipsius x'. Et sic de aliis.

Cor. Vi hujus Definitionis, $x' + \underset{\cdot}{x}' = x$, $x + \underset{\cdot}{x} = \underset{\prime}{x}$, $\underset{\cdot}{x} + \underset{\cdot\cdot}{x} = \underset{\prime\cdot}{x}$. Et sic de aliis hujusmodi.

Quando usu venit ut variabilis quantitas, puta x, spectanda sit tanquam Incrementum, ejus Integrale designo literâ inter uncos [ ] inclusâ. Istius etiam Integralis $[x]$ Integrale (vel ipsius x Integrale secundum,) designo numero binario uncorum priori superimposito, ut $\overset{2}{[}x]$. Istius etiam Integralis Integrale (vel ipsius x Integrale tertium,) ad eundem modum designo numero ternario, ut $\overset{3}{[}x]$. Et sic deinceps. Unde vi hujus Definitionis constituunt $\overset{3}{[}x]$, $\overset{2}{[}x]$, $[x]$, x Seriem terminorum, quorum quilibet est ipsum immediatè præcedentis incrementum primum, ut sit $\overset{2}{[}x] = \overset{3}{[}\underset{\cdot}{x}]$, $[x] = \overset{2}{[}\underset{\cdot}{x}]$, $x = [\underset{\cdot}{x}]$.

Lemma.

Facti $x\,v$ ex Multiplicatione duorum variabilium x & v, incrementum est $\underset{\cdot}{x}\,v + \underset{\prime}{x}\,\underset{\cdot}{v}$.

Nam auctis variabilibus per propria incrementa, fit novum productum $\overline{x + \underset{\cdot}{x}} \times \overline{v + \underset{\cdot}{v}}$, sive $x\,v + \underset{\cdot}{x}\,v + \overline{x + \underset{\cdot}{x}} \times \underset{\cdot}{v}$, hoc est $x\,v + \underset{\cdot}{x}\,v + \underset{\prime}{x}\,\underset{\cdot}{v}$ (pro $x + \underset{\cdot}{x}$ scripto $\underset{\prime}{x}$ *per Def.* 1.)

Unde dempto priori producto $x\,v$, restat Incrementum $\underset{\cdot}{x}\,v + \underset{\prime}{x}\,\underset{\cdot}{v}$.

Prop. I. *Theor.*

Ejusdem Facti xv Incrementum, vel primum, vel secundum, vel tertium, vel aliud quodvis, cujus ordo designatur per symbolum n, exhibetur per formulam hanc generalem

$$\underset{n}{x}v + n\underset{\prime\,n-1}{x}\underset{\cdot}{v} + n\times\frac{n-1}{2}\underset{\prime\prime\,n-2}{x}\underset{\cdot\cdot}{v} + n\times\frac{n-1}{2}\times\frac{n-2}{3}\underset{\prime\prime\prime\,n-3}{x}\underset{\therefore}{v} + \&c.$$

In hâc formulâ hæc sunt observanda, 1^{mo} Terminorum numeri coefficientes 1, n, $n\times\frac{n-1}{2}$, $n\times\frac{n-1}{2}\times\frac{n-2}{3}$ *&c.* iidem sunt ac in binomii dignitate n. 2^{do} Numeri n, $n-1$, $n-2$, $n-3$, *&c.* ipsis x infrascripti designant numeros punctorum quibus definiuntur Incrementa. 3^{tio} Lineolæ $_{\prime}$, $_{\prime\prime}$, $_{\prime\prime\prime}$, *&c.* ipsis x infrascriptæ, interpretandæ sunt per Def. 1. 4^{to}. In quovis Termino numerus punctorum ipsis x & v simul infrascriptorum, est n. Sit *v.g.* $n=4$: tum per formulam, ipsius xv incrementum quartum prodit $\underset{::}{x}v + 4\underset{\prime\,\therefore}{x}\underset{\cdot}{v} + 6\underset{\prime\prime\,\cdot\cdot}{x}\underset{\cdot\cdot}{v} + 4\underset{\prime\prime\prime\,\cdot}{x}\underset{\therefore}{v} + \underset{\prime\prime\prime\prime}{x}\underset{::}{v}$.

Theorema hoc generale demonstrari potest per Inductionem, incrementis continuò sumptis juxta formam in Lemmate præcedenti traditam. Sed & collectâ formâ Seriei ex hujusmodi calculo, Theorema etiam demonstrari potest per Methodum Incrementorum, ad eum modum cujus specimen mox dabimus in demonstratione Propositionis tertiæ.

Prop. II. *Theor.*

Ipsius xv Integrale primum $[xv]$ exhibetur per Seriem $[x]v - \overset{2}{[\underset{\prime}{x}]}\underset{\cdot}{v} + \overset{3}{[\underset{\prime\prime}{x}]}\underset{\cdot\cdot}{v} - \overset{4}{[\underset{\prime\prime\prime}{x}]}\underset{\therefore}{v}$ *&c.*

Series autem ita terminatur, ut sit $[xv] = [x]v$

$$-\left[[\underset{\prime}{x}]\underset{\cdot}{v}\right] = [x]\,v - \overset{2}{[\underset{\prime}{x}]}\underset{\cdot}{v} + \left[\overset{2}{[\underset{\prime\prime}{x}]}\underset{\cdot\cdot}{v}\right] = \&c.$$

Nam ſumendo incrementa reſtituitur propoſitum xv.

Cor. 1. Datis duobus ex iſtis $[x]$, $[xv]$, $\left[[\underset{\prime}{x}]\underset{\cdot}{v}\right]$, datur tertium. Item datis tribus ex iſtis $[x]$, $\overset{2}{[\underset{\prime}{x}]}$, $[xv]$, $\left[\overset{2}{[\underset{\prime\prime}{x}]}\underset{\cdot}{v}\right]$, datur quartum, Et ſic porrò.

Cor. 2. Si $\underset{\cdot}{v} = 0$, datur $[xv]$ ex dato $[x]$. Si $\underset{\cdot\cdot}{v} = 0$ datur $[xv]$ ex datis duobus $[x]$, & $\overset{2}{[x]}$, Si $\underset{\cdot\cdot\cdot}{v} = 0$, datur $[xv]$, ex datis tribus $[x]$, $\overset{2}{[x]}$, $\overset{3}{[x]}$. Et ſic porrò.

Ex. 1. Sit exemplum hujus formulæ in inventione Integralis ipſius $\dfrac{v}{z\underset{\prime}{z}\underset{\prime\prime}{z}\underset{\prime\prime\prime}{z}}$, dato nempe $\underset{\cdot}{z}$, atque exiſtente $\underset{\cdot\cdot\cdot}{v} = 0$, qui caſus eſt ſpecialis Propoſitionis ſecundæ Tractatus præcedentis D^ni^ *Monmort*. Facto itaque $x = \dfrac{1}{z\underset{\prime}{z}\underset{\prime\prime}{z}\underset{\prime\prime\prime}{z}}$, ſunt $[x] = \dfrac{-1}{3\underset{\cdot}{z}z\underset{\prime}{z}\underset{\prime\prime}{z}}$, $\overset{2}{[\underset{\prime}{x}]} = \dfrac{1}{2\underset{\cdot}{z}\times 3\underset{\cdot}{z}z\underset{\prime}{z}\underset{\prime\prime}{z}}$, atque $\overset{3}{[\underset{\prime\prime}{x}]} = \dfrac{-1}{1\underset{\cdot}{z}\times 2\underset{\cdot}{z}\times 3\underset{\cdot}{z}z\underset{\prime\prime}{z}}$. Unde per formulam fit $[xv]$, hoc eſt $\left[\dfrac{v}{z\underset{\prime}{z}\underset{\prime\prime}{z}\underset{\prime\prime\prime}{z}}\right] = -\dfrac{v}{3\underset{\cdot}{z}z\underset{\prime}{z}\underset{\prime\prime}{z}} =$

$$-\frac{\underset{\cdot}{v}}{2\underset{\cdot}{z}\times 3\underset{\cdot}{z}z\underset{\prime}{z}\underset{\prime\prime}{z}} - \frac{\underset{\cdot\cdot}{v}}{1\underset{\cdot}{z}\times 2\underset{\cdot}{z}\times 3\underset{\cdot}{z}z\underset{\prime\prime}{z}}.$$

Ex.

Ex. 2. Sit aliud exemplum in inventione Integralis ipsius $n a^z$, ubi est $\dot{z} = 1$, atque datur a, Tum pro x sumpto a^z, & pro v sumpto n, fit $\overline{x = a^{\dot{z}}}$ hoc est $\overline{x} = a x$, seu $x + \dot{x} = a x$, adeoque $\dot{x} = \overline{a - 1}\, x$,

atque $x = \frac{\dot{x}}{a - 1}$. Regrediendo itaque ad Integralia fit

$[x] = \frac{x}{a - 1}$; item $\overset{2}{[x]} = \frac{[x]}{a - 1} = \frac{x}{\overline{a - 1}|^2}$, item $\overset{3}{[x]} = \frac{x}{\overline{a - 1}|^3}$; & sic porrò. Adeoque (quoniam $\underset{\prime}{x} = a x$,) sunt

$[x] = \frac{x}{a - 1}$, $[\underset{\prime}{x}] = \frac{a x}{\overline{a - 1}|^2}$, $[\underset{\prime\prime}{x}] = \frac{a^2 x}{\overline{a - 1}|^3}$, &c. Unde

per formulam prodit $[n a^z] = \frac{a^z n}{a - 1} - \frac{a^{z+1} \dot{n}}{\overline{a - 1}|^2} + \frac{a^{z+2} \ddot{n}}{\overline{a - 1}|^3}$ &c.

In hoc exemplo continetur Solutio Problematis, de quo agit D^{nus} de *Monmort* in Propositione nona. Coincidit autem formula cum ea quam exhibet ille in Corollario primo ejusdem Propositionis.

Scholium. Possunt etiam ex hâc formulâ alii derivari valores Integralis quæsiti, pro vario modo quo interpretantur Incrementi propositi factores. Sic in exemplo secundo integrale ipsius $n a^z$ exhiberi potest per formulam $a^z [n] - \overline{a - 1}\, a^z \overset{2}{[\underset{\prime}{n}]} + \overline{a - 1}|^2\, a^z \overset{3}{[\underset{\prime\prime}{n}]}$ — &c. pro x nempe sumpto n, & pro v sumpto a^z. Sed de his fortasse aliâ occasione fusius dicemus.

Prop. III. *Theor.*

Ejusdem $x v$ Integrale, vel primum, vel secundum, vel tertium, vel aliud quodvis cujus ordo designatur symbolo n, exhibetur per Seriem in hâc formâ generali prodeuntem $\overset{n}{[x v]} = \overset{n}{[x]}\, v - n \overset{n+1}{[\underset{\prime}{x}]}\, \dot{v} +$

$+ n\, x$

$+ n \times \frac{n+1}{2} \overset{n+2}{[\underset{\prime\prime}{x}]} \underset{\cdot\cdot}{v} - n \times \frac{n+1}{2} \times \frac{n+2}{3} \overset{n+3}{[\underset{\prime\prime\prime}{x}]} \underset{\therefore}{v} + \&c.$

Collectâ formâ Seriei ex Propositione præcedenti, Coefficientes $1, - n, n \times \frac{n+1}{2}, - n \times \frac{n+1}{2} \times \frac{n+2}{3}, \&c.$ sic inveniuntur per Methodum Incrementorum. Pone $\overset{n}{[xv]} = A \overset{n}{[x]} v + B \overset{n+1}{[\underset{\prime}{x}]} \underset{\cdot}{v} + C \overset{n+2}{[\underset{\prime\prime}{x}]} \underset{\cdot\cdot}{v} + D \overset{n+3}{[\underset{\prime\prime\prime}{x}]} \underset{\therefore}{v} + \&c.$

Tum aucto n incremento suo $\underset{\cdot}{n} = 1$, atque ipsis A, B, C, D, &c. incrementis suis contemporaneis $\underset{\cdot}{A}, \underset{\cdot}{B}, \underset{\cdot}{C}, \underset{\cdot}{D}$, &c. ut jam evadant $\underset{\prime}{n}, \underset{\prime}{A}, \underset{\prime}{B}, \underset{\prime}{C}, \underset{\prime}{D}$, &c. fiet novum Integrale (quod Integrale est ipsius $\overset{n}{[xv]}$,) $\overset{n+1}{[xv]} = \underset{\prime}{A} \overset{n+1}{[x]} v + \underset{\prime}{B} \overset{n+2}{[\underset{\prime}{x}]} \underset{\cdot}{v} + \underset{\prime}{C} \overset{n+3}{[\underset{\prime\prime}{x}]} \underset{\cdot\cdot}{v} + \underset{\prime}{D} \overset{n+4}{[\underset{\prime\prime\prime}{x}]} \underset{\therefore}{v} + \&c.$ Hujus itaque Incrementum primum coincidere debet cum Integrali prius posito. Sumptis ergo incrementis, fit

$$\overset{n}{[xv]} = \underset{\prime}{A} \overset{n}{[x]} v \begin{matrix} + \underset{\prime}{A} \\ + \underset{\prime}{B} \end{matrix} \overset{n+1}{[\underset{\prime}{x}]} \underset{\cdot}{v} \begin{matrix} + \underset{\prime}{B} \\ + \underset{\prime}{C} \end{matrix} \overset{n+2}{[\underset{\prime\prime}{x}]} \underset{\cdot\cdot}{v} \begin{matrix} + \underset{\prime}{C} \\ + \underset{\prime}{D} \end{matrix} \overset{n+3}{[\underset{\prime\prime\prime}{x}]} \underset{\therefore}{v} +$$

idem ac Integrale prius positum. Itaque terminos homologos inter se comparando fit 1mo $\underset{\prime}{A} = A$. Unde est A datum quid. Sed ubi $n = 0$, est $A = 1$, ergo $A = 1$. 2do. $B = \underset{\prime}{B} + \underset{\prime}{A}$, hoc est $B = B + \underset{\cdot}{B} + 1$, seu $\underset{\cdot}{B} = - 1 = - \underset{\cdot}{n}$. Ergo regrediendo ad Integralia, fit $B = - n + a$. Sed ubi $n = 0$, est $B = 0$. Ergo $a = 0$, atque $B = - n$. 3tio. $C = \underset{\prime}{C} + \underset{\prime}{B}$, hoc est $\underset{\cdot}{C} = \underset{\prime}{n}$. Regre-

diendo itaque ad Integralia fit $C = \frac{n\,\underset{\prime}{n}}{2} + b$. Sed ubi $n = 0$, est $C = 0$. Ergo $b = 0$, atque $C = \frac{n\,\underset{\prime}{n}}{2}$, hoc est, $n \times \frac{n+1}{2}$. 4to. Ad eundem modum invenitur $D = -n \times \frac{n+1}{2} \times \frac{n+2}{3}$. Et sic pergendo inveniuntur cæteri Coefficientes.

Scholium. 1. In hâc Propositione comparatâ cum Propositione primâ, cernitur singularis quædam relatio Incrementa inter & Integralia. Ut enim in Arithmeticâ vulgari, Multiplicatio & Divisio sunt invicem ita contrariæ, ut si Multiplicatio designetur per Indicem affirmativum, Divisio designabitur per Indicem cum signo negativo; sic etiam in Methodo Incrementorum, si Incrementum designetur per Indicem affirmativum, Index negativus Integrale sistet. Sic in Propositione primâ, si pro n sumatur Numerus binarius 2, per formulam exhibebitur ipsius xv incrementum secundum, nempe $\underset{\cdot\cdot}{x}\,v + 2\,\underset{\prime\,\cdot}{x}\,\underset{\cdot}{v} + \underset{\prime\prime}{x}\,\underset{\cdot\cdot}{v}$; Sed si pro n sumatur numerus negativus -2, ut jam quæratur ipsius xv incrementum (ita loqui liceat) negativè secundum, (quod idem est ac Integrale secundum) prodeunt coefficientes iidem ac si sumatur n affirmativè in Propositione præsenti: atque interpretatis insuper ipsis $\underset{-2}{x}$, $\underset{\prime -3}{x}$, $\underset{\prime\prime -4}{x}$, &c. per $\overset{2}{[x]}$, $\overset{3}{[\underset{\prime}{x}]}$, $\overset{4}{[\underset{\prime\prime}{x}]}$, &c. Series fit omnino eadem ac per Propositionem præsentem prodit, ubi quæritur Integrale secundum.

2. Ex his autem formulis quasi suâ sponte procedunt formulæ Propositionum undecimæ atque duodecimæ Libri de Methodo Incrementorum. Nam pro incre-

incrementis scribe Fluxiones, atque evanescentibus incrementis fiant jam omnes $\overset{\prime}{x}$, $\overset{\prime\prime}{x}$, $\overset{\prime\prime\prime}{x}$, x, *&c.* inter se æquales, atque migrabit statim hæc Propositio secunda in illam undecimam, atque præsens tertia in illam duodecimam. Quod quidem exemplum satis insigne est Methodi *Newtonianæ*, quâ colligit ille rationes Fluxionum ex rationibus ultimis Incrementorum evanescentium, vel ex primis nascentium.

Additamentum.

PRæcedentium impressioni intentus dum Typothetarum erroribus corrigendis do operam, atque eâ occasione in animo illa sæpius revolvo, subiit Artificium illud quo jam olim usus est D. *Jac Bernoulli* in inventione quarundam Serierum, ope Progressionis Harmonicæ cujus meminit D de *Monmort* in *Scholio* 6. *Prop.* V. præcedente commodè etiam applicari posse ad inventionem ipsius *Monmortii* Propositionum $2^{dæ}$, $3^{tiæ}$, $4^{tæ}$, $5^{tæ}$, atque id genus aliarum aliquanto fortasse generaliorum. Hoc in sequentibus paucis ostendisse, credebam Lectori non fore ingratum.

Theorema.

Sit Progressio Arithmetica p, $p+n$, $p+2n$, *&c.* cujus termini singuli successivè designentur per x, & sunto b, c, d, *&c.* quivis multiplices differentiæ datæ n terminorum Progressionis istius Arithmeticæ. Sint A, B, C, D, *&c.* Numeri quilibet dati, & constituantur fractiones quotvis $\frac{A}{x}$, $\frac{B}{x+b}$, $\frac{C}{x+c}$, $\frac{D}{x+d}$, *&c.* Pro x successive scriptis valoribus suis p, $p+n$, $p+2n$, *&c.*

ex harum fractionum quâlibet, oritur Series Harmonicè proportionalium Sic *v g.* ex fractione primâ $\frac{A}{p}$, oritur Series $\frac{A}{p}$, $\frac{A}{p+n}$, $\frac{A}{p+2n}$, *&c.* Dico quod aggregatum quotlibet hujusmodi Serierum in infinitum continuatarum in terminis numero finitis exhiberi potest, si modo fuerit numeratorum *A, B, C, D, &c.* aggregatum æquale nihilo. Duobus exemplis hoc fiet manifestum.

Ex. Sint duæ tantùm fractiones $\frac{A}{x}$, atque $\frac{-A}{x+3n}$, existente $b = 3n$. Scribantur Series harmonicæ ex his formulis ortæ, eo ordine, ut termini, in quibus sunt denominatores æquales, sibi invicem respondeant, & collectis summis terminorum homologorum, prodibit aggregatum Serierum in terminis numero finitis, ut in calculo apposito videre est.

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A}{p+2n}+\frac{A}{p+3n}+\frac{A}{p+4n}+\&c. = \text{Seriei ortæ ex } \frac{A}{x}$$

$$+\frac{-A}{p+3n}+\frac{-A}{p+4n}+\&c. = \text{Seriei ex } \frac{-A}{x+3n}$$

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A}{p+2n}+\ 0\ +\ 0\ +\&c. = \text{Aggreg. Serierū.}$$

Ex. 2. Sint tres fractiones $\frac{A}{x}$, $\frac{B}{x+2n}$, $\frac{C}{x+3a}$, existentibus $b=2n$, $c=3n$, atque $A+B+C=0$. In hoc casu Calculus sic se habet.

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A}{p+2n}+\frac{A}{p+3n}+\ldots+\&c. = \text{Seriei ortæ ex } \frac{A}{x}$$

$$+\frac{B}{p+2n}+\frac{B}{p+3n}+\ldots+\&c. = \text{Seriei ex } \frac{B}{x+2n}$$

$$+\frac{C}{p+3n}+\ldots+\&c. = \text{Seriei ex } \frac{C}{x+3n}$$

$$\frac{A}{p}+\frac{A}{p+n}+\frac{A+B}{p+2n}+\frac{A+B+C=0}{p+3n}+\&c. = \text{Aggregato Serierum.}$$

Ubi

Ubi etiam prodit aggregatum Serierum in terminis numero finitis, nempe $\frac{A}{p}+\frac{A}{p+n}+\frac{A+B}{p+2n}$, ob Numeratorum A, B, C, aggregatum æquale nihilo. Et ad eundem modum demonſtratur Theorema in aliis caſibus quibuſvis.

Cor. 1. Ex his principiis derivari poſſunt innumeræ Series in infinitum continuatæ, in terminis tamen numero finitis ſummabiles.

Caſ. 1. Sint $\frac{A}{x}$ & $\frac{-A}{x+b}$ formulæ duarum Serierum harmonicarum quarum aggregatum prodit in terminis numero finitis per ſuperius demonſtrata, Tum, formulis iſtis in unam ſummam collectis, fit $\frac{Ab}{x\times\overline{x+b}}$ formula Seriei ſummabilis. Sint *v. gr.* $A=\frac{1}{6}$, $p=1$, $n=2$, atque $b=3n=6$. Tum formulæ Serierum harmonicarum erunt $\frac{1}{6x}$, & $\frac{-1}{6\times\overline{x+6}}$, formula Seriei compoſitæ ſummabilis erit $\frac{1}{x\times\overline{x+6}}$, Serie illa exiſtente $\frac{1}{1\times 7}+\frac{1}{3\times 9}+\frac{1}{5\times 11}+\frac{1}{7\times 13}+$ &c. atque ſumma Seriei, per calculum in præmiſſis demonſtratum, erit $\frac{1}{6\times 1}+\frac{1}{6\times 3}+\frac{1}{6\times 5}$. Sint tres formulæ Serierum harmonicarum $\frac{A}{x}$, $\frac{B}{x+b}$, $\frac{C}{x+c}$, (exiſtente $A+B+C=0$, ut ſit Serierum aggregatum finitum per præmiſſa.) Tum formulis in unam ſummam collectis fit

$$\frac{A\times\overline{x+b}\times\overline{x+c}+B\times x\times\overline{x+c}+C\times x\times\overline{x+b}}{x\times\overline{x+b}\times\overline{x+c}},$$

ſeu (terminis revocatis ad formam factorum x, $x\times\overline{x+b}$, $x\times\overline{x+b}\times\overline{x+c}$,)

$$\frac{Acb+\overline{Ac+\overline{c-b}\,B}\times x+\overline{A+B+C}\times x\times\overline{x+b}}{x\times\overline{x+b}\times\overline{x+c}},$$

hoc eſt

(ob $A + B + C = 0$) $\frac{Acb + \overline{Ac + B} \times \overline{c - b} \times x}{x \times \overline{x + b} \times \overline{x + c}}$, formula Seriei ſummabilis. Si quatuor ſint Fractiones $\frac{A}{x}$, $\frac{B}{x + b}$, $\frac{C}{x + c}$, $\frac{D}{x + d}$, (exiſtente $A + B + C + D = 0$) ad eundem modum invenietur formula Seriei ſummabilis

$$\frac{Abcd + \overline{Acd + B \times \overline{c - b} \times \overline{d - b}} \times x + \overline{Ad + B \times \overline{d - b} + C \times \overline{d - c}} \times x \times x + b}{x \times \overline{x + b} \times \overline{x + c} \times \overline{x + d}}$$

Et ſic pergere licet ad formulas adhuc magis compoſitas.

Caſ. 2. Et ſi plures ſint formulæ Serierum hujuſmodi ſummabilium, quarum denominatorum factores excerpantur ex diverſis progreſſionibus Arithmeticis, ex iſtarum formularum quotvis in unam ſummam additione, conficietur formula nova Seriei ſummabilis: Sint *e. gr.* formulæ duæ Serierum ſummabilium $\frac{1}{x \times \overline{x + 3}}$ & $\frac{1}{z \times \overline{z + 2}}$, excerptis x ex Progreſſione Arithmeticâ 1, 2, 3, 4, *&c.* z ex Progreſſione Arithmeticâ 1, 3, 5, *&c.* Tum ex his formulis in unam ſummam collectis fiet formula nova $\frac{z \times \overline{z + 2} + x \times \overline{x + 3}}{x \times \overline{x + 3} \times z \times \overline{z + 2}}$, vel, (expoſito z per x & numeros datos) $\frac{\overline{2x - 1} \times \overline{2x + 1} + x \times \overline{x + 3}}{x \times \overline{x + 3} \times \overline{2x - 1} \times \overline{2x + 1}}$

Cor. 2. Hinc omnis Series in infinitum continuata ſummabilis eſt, cujus termini deſignantur per Fractionem, cujus denominatoris factores excerpuntur ex datâ quâlibet Progreſſione Arithmeticâ, numerator autem eſt multinomium, cujus dimenſiones ſunt ad minimum binario pauciores, quam ſunt dimenſiones Denominatoris. Nam omnis hujuſmodi fractio reſolvi poteſt in tot fractiones ſimplices, quot ſunt dimenſiones (hoc eſt, quot ſunt factores) Denominatoris, quarum numeratorum aggregatum eſt nihil. Sit exempli gratiâ, formula

formula oblata $\frac{\alpha+\beta x+\gamma x\times\overline{x+b}}{x\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}}$. Pone hanc formulam æquari aggregato fractionum $\frac{A}{x}+\frac{B}{x+b}+\frac{C}{x+c}$ $+\frac{D}{x+d}$. Tum fractionibus istis in unam summam collectis fiet $Abcd+\overline{Acd+B\overline{c-b}\times\overline{d-b}}\times x$ $+\overline{Ad+B\times\overline{d-b}+C\times\overline{d-c}}\times x\times\overline{x+b}$ $+\overline{A+B+C+D}\times x\times\overline{x+b}\times\overline{x+c}$ applicatum ad $x\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}=\frac{\alpha+\beta x+\gamma x\times\overline{x+b}}{x\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}}$.

Unde per comparationem terminorum homologorum fit $Abcd=\alpha$, $Acd+B\times\overline{c-b}\times\overline{d-b}=\beta$, $Ad+B\times\overline{d-b}+C\times\overline{d-c}=\gamma$, $A+B+C+D=0$. adeoque $A=\frac{\alpha}{bcd}$, $B=\frac{\beta-Acd}{\overline{c-b}\times\overline{d-b}}$,

$C=\frac{\gamma-Ad-B\times\overline{d-b}}{d-c}$, $D=-A-B-C$, Quo pacto formula oblata resolvitur in fractiones simplices $\frac{\alpha}{bcdx}$ $+\frac{\beta-Acd}{\overline{c-b}\times\overline{d-b}\times\overline{x+b}}+\frac{\gamma-Ad-B\times\overline{d-b}}{\overline{d-c}\times\overline{x+c}}$ $+\frac{-A-B-C}{x+d}$, ex quibus ortarum Serierum aggregatum, hoc est, summa Seriei ortæ ex formulâ oblatâ $\frac{\alpha+\beta x+\gamma x\times\overline{x+b}}{x\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}}$, per jam dicta prodit in terminis numero finitis. Quod verò dimensiones numeratoris in formulâ oblatâ, debeant esse binario ad minimum pauciores, quam sunt dimensiones Denominatoris, hinc constat quod in reductione fractionum $\frac{A}{x}$, $\frac{B}{x+b}$, $\frac{C}{x+c}$, $\frac{D}{x+d}$, quilibet numerator A, B, C, D,

ducitur

ducitur in omnes denominatores excepto uno, nempe suo; unde prodeunt Numeratoris Dimensiones unitate pauciores quam sunt dimensiones Denominatoris. Sed per æquationem $A + B + C + D = 0$ perit altissima dimensio in numeratore; Unde supersunt Numeratoris Dimensiones ad minimum binario pauciores quam sunt dimensiones Denominatoris. Ad hoc verò Corollarium revocari possunt D. de *Monmort* Propositiones 2da & 5ta.

Cor. 3. Item oblatâ formulâ juxta *Cas.* 2. *Cor.* 1. adhuc magis compositâ, ex iidem principiis perspici potest an sit Series summabilis. Sint progressiones duæ Arithmeticæ 1, 3, 5, *&c.* 2, 4, 6, *&c.* quarum termini homologi designentur per x & z, & sit formula Seriei oblata $\frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+2} \times z \times \overline{z+2}}$, vel (pro z scripto $x+1$, & factoribus Denominatoris in ordinem coactis) $\frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}$. Pone formulam hanc æquari aggregato formularum $\frac{P}{x \times \overline{x+2}}$, $\frac{Q}{\overline{x+1} \times \overline{x+3}}$, Serierum per superius dicta summabilium, ut (formulis his novissimis in unam summam collectis) sit $\frac{P \times \overline{x+1} \times \overline{x+3} + Q \times x \times \overline{x+2}}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}$ seu $\frac{3P + \overline{4P + 2Q}\,x + \overline{P+Q}\,x^2}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}} = \frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}$. Hinc comparando terminos homologos oriuntur æquationes $3P = \alpha$, $4P + 2Q = \beta$, $P + Q = \gamma$. Unde eliminatis P & Q per debitas operationes Analyticas, prodit æquatio $2\alpha - 3\beta + \gamma = 0$, qua definitur relatio quæ inter coefficientes α β, γ intercedere debet, ut Series orta ex formulâ oblatâ $\frac{\alpha + \beta x + \gamma x^2}{x \times \overline{x+1} \times \overline{x+2} \times \overline{x+3}}$ sit

fit fummabilis. Ad eundem modum fi formulæ oblatæ Denominatoris factores excerpantur ex tribus Progreffionibus Arithmeticis, invenientur duæ æquationes quibus definiantur relationes coefficientium Numeratoris, ut fit Series fummabilis. Si quatuor fint Progreffiones Arithmeticæ, Coefficientium relatio definietur per tres æquationes. Et fic porrò. Et in hujufmodi formulis ut fint Series fummabiles, hæc infuper obfervanda funt, Primò ut Numeratorum dimenfiones fint ad minimum binario pauciores quam funt dimenfiones Denominatorum, Deinde ut ex fingulis Progreffionibus Arithmeticis excerpantur ad minimum duo factores Denominatoris. Denique, quod fi fint duo vel plures factores Denominatoris inter fe æquales, ponendum fit tot etiam Progreffiones Arithmeticas, ex quibus excerpuntur, effe inter fe æquales. Præmiffis attentius perpenfis, hæc obvia erunt. Ad hoc vero Corollarium facile revocantur D. de *Monmort* Propofitiones 3^tia^ & 4^ta^.

FINIS.

ERRATUM in N°. 352.

PAge 586, after the end of line 15, add *black Cloud, from behind which there iffued a.*

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